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On Galois coverings and tilting modules

Patrick Le Meur

CMLA, ENS Cachan, CNRS, UniverSud, 61 Avenue du President Wilson, F-94230 Cachan, France

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Abstract

Let A be a basic connected finite dimensional algebra over an algebraically closed field k. Let T be a basic tilting A-module with arbitrary finite projective dimension. For a fixed group G we compare the set of isoclasses of connected Galois coverings of A with group G and the set of isoclasses of connected Galois coverings of $\mathrm{End}_A(T)$ with group G. Using the Hasse diagram $\vec{\mathcal{K}}_A$ (see [D. Happel, L. Unger, On a partial order of tilting modules, Algebr. Represent. Theory 8 (2) (2005) 147–156] and [C. Riedtmann, A. Schofield, On a simplicial complex associated with tilting modules, Comment. Math. Helv. 66 (1) (1991) 70–78]) of basic tilting A-modules, we give sufficient conditions on T under which there is a bijection between these two sets (these conditions are always verified when A is of finite representation type). Then we apply these results to study when the simple connectedness of A implies the one of $\mathrm{End}_A(T)$ (see [I. Assem, E.N. Marcos, J.A. de la Peña, The simple connectedness of a tame tilted algebra, J. Algebra 237 (2) (2001) 647–656]). Finally, using an argument due to W. Crawley-Boevey, we prove that the type of any simply connected tilted algebra is a tree and that its first Hochschild cohomology group vanishes.

Keywords: Finite dimensional algebra; Representation theory; Tilting theory; Galois covering; Fundamental group; Simple connectedness

Introduction

Let k be an algebraically closed field and let A be a finite dimensional k-algebra. In order to study the category $\mathrm{mod}(A)$ of finite dimensional (left) A-modules we may assume that A is basic and connected. In the study of $\mathrm{mod}(A)$, tilting theory has proved to be a powerful tool. Indeed, if T is a basic tilting A-module and if we set $B = \mathrm{End}_A(T)$, then A and B have many

E-mail address: patrick.lemeur@cmla.ens-cachan.fr.

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common properties: Brenner–Butler Theorem establishes an equivalence between certain subcategories of $\mathrm{mod}(A)$ and $\mathrm{mod}(B)$ (see [10,17] and [22]), A and B have equivalent derived categories (see [15]) and (in particular) they have isomorphic Grothendieck groups and isomorphic Hochschild cohomologies. In this text we study the following problem relating A and B:

Is it possible to compare the Galois coverings of A and those of B? (P_1)

As an example, if $A = kQ$ with Q a finite quiver without oriented cycle and if T is an APR-tilting module associated to a sink x of Q (see [6]) then $B = kQ'$ where Q' is obtained from Q by reversing all the arrows endings at x. In particular Q and Q' have the same underlying graph and therefore A has a connected Galois covering with group G if and only if the same holds for B.

Recall that in order to consider Galois coverings of A we always consider A as a k-category. When $\mathcal{C} \to A$ is a Galois covering, it is possible to describe part of $\mathrm{mod}(A)$ in terms of $\mathcal{C}$-modules (see for example [9] and [13]). This description is useful because $\mathrm{mod}(\mathcal{C})$ is easier to study than $\mathrm{mod}(A)$, especially when $\mathcal{C}$ is simply connected (this last situation may occur when A is of finite representation type, see [13]). Notice that simple connectedness and tilting theory have already been studied together through the following conjecture formulated in [5]:

If A is simply connected, then B is simply connected. (P_2)

More precisely, the above implication is true if: A is of finite representation type and T is of projective dimension at most one (see [5]), or if: $A = kQ$ (with Q a quiver) and B is tame (see [3], see also [1] for a generalisation to the case of quasi-tilted algebras). The two problems (P_1) and (P_2) are related because A is simply connected if and only if there is no proper Galois covering $\mathcal{C} \to A$ with $\mathcal{C}$ connected and locally bounded (see [20]).

In order to study the question (P_1) we will exhibit sufficient conditions for T to be of the first kind with respect to a fixed Galois covering $\mathcal{C} \xrightarrow{F} A$. Indeed, if T is of the first kind with respect to F, then it is possible to construct a Galois covering of B. Under additional hypotheses on T, the equivalence class of this Galois covering is uniquely determined by the equivalence class of F. Here we say that two Galois coverings $F : \mathcal{C} \to A$ and $F' : \mathcal{C}' \to A$ are equivalent if and only if there exists a commutative square of k-categories and k-linear functors:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\sim"] \arrow[d, "F"'] & \mathcal{C}' \arrow[d, "F'"] \\ A \arrow[r, "\sim"] & A, \end{tikzcd}$$

where horizontal arrows are isomorphisms and where the bottom horizontal arrow restricts to the identity map on the set of objects of A. For simplicity, let us say that A and B have the same connected Galois coverings with group G if there exists a bijection between the sets $\mathrm{Gal}_A(G)$ and $\mathrm{Gal}_B(G)$ where $\mathrm{Gal}_A(G)$ and $\mathrm{Gal}_B(G)$ stand for the set of equivalence classes of Galois coverings $\mathcal{C} \to A$ and $\mathcal{C} \to B$ respectively, with group G and with $\mathcal{C}$ connected and locally bounded. With this definition, we prove the following theorem which is the main result of this text and which partially answers (P_1):

Theorem 1. *Let T be a basic tilting A-module, let $B = \mathrm{End}_A(T)$ and let G be a group.*

1. *If $T' \in \vec{\mathcal{K}}_A$ lies in the connected component of $\vec{\mathcal{K}}_A$ containing T, then $\mathrm{End}_A(T)$ and $\mathrm{End}_A(T')$ have the same connected Galois coverings with group G.*
2. *If T lies in the connected component of $\vec{\mathcal{K}}_A$ containing A or DA, then A and B have the same connected Galois coverings with group G.*

In particular, if $\vec{\mathcal{K}}_A$ is connected (which happens when A is of finite representation type) then A and B have the same connected Galois coverings with group G.

Here $\vec{\mathcal{K}}_A$ is the Hasse diagram associated with the poset $\mathcal{T}_A$ of basic tilting A-modules (see [18] and [24]). Recall (see [13]) that when A is of finite representation type and standard, A admits a connected Galois covering with group G if and only if G is a factor group of the fundamental group $\pi_1(A)$ of the Auslander–Reiten quiver of A with its mesh relations. Theorem 1 allows us to get the following corollary when A and B are of finite representation type and standard. We thank Ibrahim Assem for having pointed out this corollary.

Corollary 1. *Let T be a basic tilting A-module and let $B = \mathrm{End}_A(T)$. If both A and B are of finite representation type and standard, then the Auslander–Reiten quivers of A and B have isomorphic fundamental groups.*

Theorem 1 also allows us to prove the following corollary related to (P_2).

Corollary 2. *(See [5] and [1].) Let T be a basic tilting A-module and let $B = \mathrm{End}_A(T)$.*

1. *If $T' \in \vec{\mathcal{K}}_A$ lies in the connected component of $\vec{\mathcal{K}}_A$ containing T, then: $\mathrm{End}_A(T)$ is simply connected if and only if $\mathrm{End}_A(T')$ is simply connected.*
2. *If T lies in the connected component of $\vec{\mathcal{K}}_A$ containing A or DA then: A is simply connected if and only if B is simply connected.*

In particular, if $\vec{\mathcal{K}}_A$ is connected (for example, A is of finite representation type, see [18]), then: A is simply connected if and only if B is simply connected.

Theorem 1 uses the Hasse diagram $\vec{\mathcal{K}}_A$ in order to prove (in particular) that a basic tilting A-module lying in a specific connected component of $\vec{\mathcal{K}}_A$ is of the first kind with respect to a given connected Galois covering of A. On the other hand, an argument due to W. Crawley-Boevey proves that any A-module M such that $\mathrm{Ext}^1_A(M, M) = 0$ is of the first kind with respect to any Galois covering of A with group $\mathbb{Z}$ (or with group $\mathbb{Z}/p\mathbb{Z}$) if $char(k) = 0$ (or if $char(k) = p$, p prime, respectively). Using this argument we are able to prove the following proposition. It gives a partial answer to (P_2) and to A. Skowroński's question in [25, Problem 1] whether it is true that a triangular algebra is simply connected if and only if its first Hochschild cohomology group vanishes.

Proposition 1. *(See [1,3].) Let B be a tilted algebra of type Q. If B is simply connected, then Q is a tree and $HH^1(B) = 0$.*

Recall that the implication of Proposition 1 has been proved in [3] for B tame tilted and in [1] for B tame quasi-tilted.

The text is organised as follows. In Section 1 we give the definition of all the notions mentioned above and which are used for the proof of Theorem 1. In Section 2 we detail the construction and give some properties of the Galois covering F' of B starting from a Galois covering $F:\mathcal{C}\to A$ of A and a basic tilting A-module T. In this study, we introduce the following hypotheses on the A-module T: (1) T is of the first kind with respect to F (this ensures that F' exists), (2) the $\mathcal{C}$-module $F.T$ obtained from T by restricting the scalars is basic (this ensures that F' is connected if F is connected), (3) $\psi.N\simeq N$ for any direct summand N of T and for any automorphism $\psi:A\xrightarrow{\sim}A$ which restricts to the identity map on objects (this ensures that the equivalence class of F' does depend only on the equivalence class of F). These three hypotheses lack of simplicity, therefore, Section 3 is devoted to find simple sufficient conditions for the basic tilting A-module T to verify these. In particular, we prove that the condition "*T lies in the connected component of $\vec{\mathcal{K}}_A$ containing A*" fits our requirements. Since our main objective is to establish a correspondence between the equivalence classes of the connected Galois coverings of A and those of B, we need to find conditions for T to lie in both connected components of $\vec{\mathcal{K}}_A$ and $\vec{\mathcal{K}}_B$ containing A and B respectively (recall that T is also a basic tilting B-module). This is done in Section 4 where we compare the Hasse diagrams $\vec{\mathcal{K}}_A$ and $\vec{\mathcal{K}}_B$. In particular, we prove that there is an oriented path in $\vec{\mathcal{K}}_A$ starting at A and ending at T if and only if there is an oriented path in $\vec{\mathcal{K}}_B$ starting at B and ending at T. This equivalence is used in Section 5 in order to prove Theorem 1, Corollaries 1 and 2. Finally, in Section 6, we prove Proposition 1.

1. Basic definitions and preparatory lemmata

Reminder on k-categories (see [9] for more details)

A *k-category* is a small category $\mathcal{C}$ such that for any $x,y\in\mathrm{Ob}(\mathcal{C})$ the set ${}_y\mathcal{C}_x$ of morphisms from x to y is a k-vector space and such that the composition of morphisms in $\mathcal{C}$ is k-bilinear. A k-category $\mathcal{C}$ is called *connected* if and only if there is no non-trivial partition $\mathrm{Ob}(\mathcal{C})=E\sqcup F$ such that ${}_y\mathcal{C}_x={}_x\mathcal{C}_y=0$ for any $x\in E$, $y\in F$.

All functors between k-categories are supposed to be k-linear. If $F:\mathcal{E}\to\mathcal{B}$ and $F':\mathcal{E}'\to\mathcal{B}$ are functors between k-categories, then F and F' are called equivalent if there exists a commutative diagram:

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\sim"] \arrow[d, "F"'] & \mathcal{E}' \arrow[d, "F'"] \\ \mathcal{B} \arrow[r, "\sim"] & \mathcal{B}, \end{tikzcd}$$

where horizontal arrows are isomorphisms and where the bottom horizontal arrow restricts to the identity map on $\mathrm{Ob}(\mathcal{B})$. A *locally bounded* k-category is a k-category $\mathcal{C}$ verifying the following conditions:

- distinct objects in $\mathcal{C}$ are not isomorphic,
- for any $x\in\mathrm{Ob}(\mathcal{C})$, the k-vector spaces $\bigoplus_{y\in\mathrm{Ob}(\mathcal{C})}{}_y\mathcal{C}_x$ and $\bigoplus_{y\in\mathrm{Ob}(\mathcal{C})}{}_x\mathcal{C}_y$ are finite dimensional,
- for any $x\in\mathrm{Ob}(\mathcal{C})$, the k-algebra ${}_x\mathcal{C}_x$ is local.

For example, let A be a basic finite dimensional k-algebra (basic means that A is the direct sum of pairwise non-isomorphic indecomposable projective A-modules) and let $\{e_1, \dots, e_n\}$ be a complete set of pairwise orthogonal primitive idempotents. Then A can be viewed as a locally bounded k-category as follows: $e_1, \dots, e_n$ are the objects of A, the space of morphisms from e_i to e_j is equal to $e_j A e_i$ for any i, j and the composition of morphisms is induced by the product in A. Notice that different choices for the primitive idempotents $e_1, \dots, e_n$ give rise to isomorphic k-categories. In this text we shall always consider such an algebra A as a locally bounded k-category.

Modules over k-categories

If $\mathcal{C}$ is a k-category, a (left) $\mathcal{C}$-module is a k-linear functor $M : \mathcal{C} \to \mathrm{MOD}(k)$ where $\mathrm{MOD}(k)$ is the category of k-vector spaces. A morphism of $\mathcal{C}$-modules $M \to N$ is a k-linear natural transformation of functors. The category of $\mathcal{C}$-modules is denoted by $\mathrm{MOD}(\mathcal{C})$.

A $\mathcal{C}$-module M is called locally finite dimensional (or finite dimensional) if and only if $M(x)$ is finite dimensional for any $x \in \mathrm{Ob}(\mathcal{C})$ (or $\bigoplus_{x \in \mathrm{Ob}(\mathcal{C})} M(x)$ is finite dimensional, respectively). The category of locally finite dimensional (or finite dimensional) $\mathcal{C}$-modules is denoted by $\mathrm{Mod}(\mathcal{C})$ (or by $\mathrm{mod}(\mathcal{C})$, respectively). Notice that if $\mathcal{C} = A$ as above, then $\mathrm{Mod}(\mathcal{C}) = \mathrm{mod}(\mathcal{C})$.

We shall write $\mathrm{IND}(\mathcal{C})$ (or $\mathrm{Ind}(\mathcal{C})$, or $\mathrm{ind}(\mathcal{C})$) for the full subcategory of $\mathrm{MOD}(\mathcal{C})$ (or of $\mathrm{Mod}(\mathcal{C})$, or of $\mathrm{mod}(\mathcal{C})$, respectively) of all the indecomposable $\mathcal{C}$-modules. Finally, if $M = N_1 \oplus \cdots \oplus N_t$ with $N_i \in \mathrm{ind}(\mathcal{C})$ for any i, then M is called *basic* if and only if $N_1, \dots, N_t$ are pairwise non-isomorphic.

Tilting modules

Let A be a basic finite dimensional k-algebra. A *tilting A-module* (see [10,17] and [22]) is a module $T \in \mathrm{mod}(A)$ verifying the following conditions:

- $(T1)$ T has finite projective dimension (that is, $\mathrm{pd}_A(T) < \infty$),
- $(T2)$ $\mathrm{Ext}^i_A(T, T) = 0$ for any $i > 0$ (that is, T is selforthogonal),
- $(T3)$ there is an exact sequence in $\mathrm{mod}(A)$: $0 \to A \to T_1 \to \cdots \to T_r \to 0$ with $T_1, \dots, T_r \in \mathrm{add}(T)$ (this last property means that $T_1, \dots, T_r$ are direct sums of direct summands of T).

Assume that T is a tilting A-module. Then, T is also a tilting $\mathrm{End}_A(T)$-module for the following action: $f.t = f(t)$ for $f \in \mathrm{End}_A(T)$ and $t \in T$. To avoid any confusion, we shall write ${}_AT$ (or ${}_BT$) when we are dealing with the A-module T (or with the B-module T, respectively). Assume moreover that T is basic as an A-module and fix a decomposition $T = T_1 \oplus \cdots \oplus T_n$ with $T_1, \dots, T_n \in \mathrm{ind}(A)$. This defines a decomposition of the identity of $\mathrm{End}_A(T)$ into a sum of primitive pairwise orthogonal idempotents so that $B := \mathrm{End}_A(T)$ is a locally bounded k-category as follows: the set of objects is $\{T_1, \dots, T_n\}$ and for any i, j the space of morphisms ${}_{T_j}B_{T_i}$ is equal to $\mathrm{Hom}_A(T_i, T_j)$. For any $x \in \mathrm{Ob}(A)$, $T(x)$ is an indecomposable B-module:

$$\begin{aligned} B &\to \mathrm{MOD}(k), \\ T_i \in \mathrm{Ob}(B) &\mapsto T_i(x), \\ u \in {}_{T_j}B_{T_i} &\mapsto T_i(x) \xrightarrow{u_x} T_j(x), \end{aligned}$$

and $T = \bigoplus_{x\in \mathrm{Ob}(A)} T(x)$. Finally, the following functor is an isomorphism of k-categories:

$$\rho_A : A \to \mathrm{End}_B(T),$$
$$x \in \mathrm{Ob}(A) \mapsto T(x) \in \mathrm{Ob}\big(\mathrm{End}_B(T)\big),$$
$$u \in {}_yA_x \mapsto T(x) \xrightarrow{T(u)} T(y).$$

For more details on the above properties and for a more general study of $\mathrm{End}_A(T)$, we refer the reader to [10,15,17] and [22].

Let $\mathcal{T}_A$ be the set of basic tilting A-modules up to isomorphism. Then $\mathcal{T}_A$ is endowed with a partial order introduced in [24] and defined as follows. If $T \in \mathcal{T}_A$, the right perpendicular category $T^\perp$ of T is defined by (see [7]):

$$T^\perp = \big\{X \in \mathrm{mod}(A) \bigm| \mathrm{Ext}^i_A(T, X) = 0 \text{ for all } i \geqslant 1\big\}.$$

If $T' \in \mathcal{T}_A$ is another basic tilting module, we write $T \leqslant T'$ provided that $T^\perp \subseteq T'^\perp$. In particular, we have $T \leqslant A$ for any $T \in \mathcal{T}_A$. In [18], D. Happel and L. Unger have proved that the Hasse diagram $\vec{\mathcal{K}}_A$ of $\mathcal{T}_A$ is as follows. The vertices in $\vec{\mathcal{K}}_A$ are the elements in $\mathcal{T}_A$ and there is an arrow $T \to T'$ in $\vec{\mathcal{K}}_A$ if and only if: $T = X \oplus \overline{T}$ with $X \in \mathrm{ind}(A)$, $T' = Y \oplus \overline{T}$ with $Y \in \mathrm{ind}(A)$ and there exists a non-split exact sequence $0 \to X \xrightarrow{u} M \xrightarrow{v} Y \to 0$ in $\mathrm{mod}(A)$ with $M \in \mathrm{add}(\overline{T})$. In such a situation, u (or v) is the minimal left (or right) $\mathrm{add}(\overline{T})$-*approximation* of X (or of Y, respectively). For more details on $\vec{\mathcal{K}}_A$, we refer the reader to [18] and [19].

Galois coverings of k-categories

Let G be a group. A *free G-category* is a k-category $\mathcal{E}$ endowed with a morphism of groups $G \to \mathrm{Aut}(\mathcal{E})$ such that the induced action of G on $\mathrm{Ob}(\mathcal{E})$ is free. In this case, there exists a (unique) quotient $\mathcal{E} \to \mathcal{E}/G$ of $\mathcal{E}$ by G in the category of k-categories. With this property, a *Galois covering with group* G of $\mathcal{B}$ is by definition a functor $F : \mathcal{E} \to \mathcal{B}$ endowed with a group morphism $G \to \mathrm{Aut}(F) = \{g \in \mathrm{Aut}(\mathcal{E}) \mid F \circ g = F\}$ and verifying the following facts:

- the group morphism $G \to \mathrm{Aut}(F) \hookrightarrow \mathrm{Aut}(\mathcal{E})$ endows $\mathcal{E}$ with a structure of free G-category,
- the functor $\mathcal{E}/G \xrightarrow{\overline{F}} \mathcal{B}$ induced by F is an isomorphism.

This definition implies that the group morphism $G \to \mathrm{Aut}(F)$ is one-to-one (actually one can show that this is an isomorphism when $\mathcal{E}$ is connected). Moreover for any $x \in \mathrm{Ob}(\mathcal{B})$ the set $F^{-1}(x)$ is non-empty and called the *fibre of* F at x. It verifies $F^{-1}(F(x)) = G.x$ for any $x \in \mathrm{Ob}(\mathcal{E})$.

We recall that Galois coverings are particular cases of *covering functors* (see [9]). A covering functor is a k-linear functor $F : \mathcal{E} \to \mathcal{B}$ such that for any $x, y \in \mathcal{E}_0$, the following mappings induced by F are bijective:

$$\bigoplus_{y' \in F^{-1}(F(y))} {}_{y'}\mathcal{E}_x \to {}_{F(y)}\mathcal{B}_{F(x)} \quad \text{and} \quad \bigoplus_{x' \in F^{-1}(F(x))} {}_y\mathcal{E}_{x'} \to {}_{F(y)}\mathcal{B}_{F(x)}.$$

Remark that a covering functor is not supposed to restrict to a surjective mapping on objects. However, a covering functor is an isomorphism of k-categories if and only if it restricts to a bijective mapping on objects. Using basic linear algebra arguments it is easy to prove the following useful lemma:

Lemma 1.1. *Let p, q be k-linear functors such that the composition $q \circ p$ is defined. If two of the functors $p, p, q \circ p$ are covering functors, then so is the third one.*

If $F : \mathcal{E} \to \mathcal{B}$ is a Galois covering with group G and with $\mathcal{B}$ connected then $\mathcal{E}$ need not be connected. In such a case, if $\mathcal{E} = \coprod_{i \in I} \mathcal{E}_i$ where the $\mathcal{E}_i$'s are the connected components of $\mathcal{E}$, then for each i, the following functor:

$$F_i : \mathcal{E}_i \hookrightarrow \mathcal{E} \to \mathcal{B}$$

is a Galois covering with group:

$$G_i := \{g \in G \mid g(\mathrm{Ob}(\mathcal{E}_i)) \cap \mathrm{Ob}(\mathcal{E}_i) \neq \emptyset\} = \{g \in G \mid g(\mathrm{Ob}(\mathcal{E}_i)) = \mathrm{Ob}(\mathcal{E}_i)\}.$$

Moreover, if $i, j \in I$ then the groups G_i and G_j are conjugate in G and there exists a commutative diagram:

$$\begin{tikzcd} \mathcal{E}_i \arrow[rr, "\sim"] \arrow[dr, "F_i"'] & & \mathcal{E}_j, \arrow[dl, "F_j"] \\ & \mathcal{B} & \end{tikzcd}$$

where the horizontal arrow is an isomorphism. This implies that G acts transitively on the set $\{\mathcal{E}_i \mid i \in I\}$ of the connected components of $\mathcal{E}$. Notice that all these facts may be false if $\mathcal{B}$ is not connected.

Two Galois coverings of $\mathcal{B}$ are called *equivalent* if and only if they are isomorphic as functors between k-categories (see above, this implies that the groups of the Galois coverings are isomorphic). The equivalence class of a Galois covering F is denoted by $[F]$. Finally, we shall say for short that a Galois covering $\mathcal{E} \to \mathcal{B}$ is *connected* if and only if $\mathcal{E}$ is connected and locally bounded (this implies that $\mathcal{B}$ is connected and locally bounded, see [13, 1.2]).

Simply connected locally bounded k-categories

Let $\mathcal{B}$ be a connected and locally bounded k-category. Then $\mathcal{B}$ is called *simply connected* if and only if there is no proper connected Galois covering of $\mathcal{B}$ (proper means with non-trivial group). This definition is equivalent to the original one (see [21] for the triangular case and [20, Corollary 4.5] for the non-triangular case) which was introduced in [4]: $\mathcal{B}$ is simply connected if and only if $\pi_1(Q_{\mathcal{B}}, I) = 1$ for any presentation $kQ_{\mathcal{B}}/I \simeq \mathcal{B}$ of $\mathcal{B}$ where $Q_{\mathcal{B}}$ is the ordinary quiver and I is an admissible ideal (see [21] for the definition of $\pi_1(Q_{\mathcal{B}}, I)$).

Basic notions on covering techniques (see [9] and [23])

Let $F:\mathcal{E}\to\mathcal{B}$ be a Galois covering with group G. The G-action on $\mathcal{E}$ gives rise to an action of G on $\mathrm{MOD}(\mathcal{E})$: if $M\in\mathrm{MOD}(\mathcal{E})$ and $g\in G$, then ${}^gM := M\circ g^{-1}\in\mathrm{MOD}(\mathcal{E})$. With this notation, we denote by G_M the *stabiliser* of M, it is the subgroup $G_M := \{g\in G \mid {}^gM\simeq M\}$ of G. Moreover, F defines two additive functors $F_\lambda:\mathrm{MOD}(\mathcal{E})\to\mathrm{MOD}(\mathcal{B})$ (the *push-down* functor) and $F_.:\mathrm{MOD}(\mathcal{B})\to\mathrm{MOD}(\mathcal{E})$ (the *pull-up* functor) with the following properties (for more details we refer the reader to [9]):

- $F_.M = M\circ F$ for any $M\in\mathrm{MOD}(\mathcal{B})$,
- if $M\in\mathrm{MOD}(\mathcal{E})$, then $(F_\lambda M)(x)=\bigoplus_{x'\in F^{-1}(x)} M(x')$ for any $x\in\mathrm{Ob}(\mathcal{B})$. If $u\in {}_y\mathcal{E}_x$, then the restriction of $(F_\lambda M)(F(u))$ to $M(g.x)$ (for $g.x\in F^{-1}(F(x))=G.x$) is equal to $M(g.u):M(g.x)\to M(g.y)$,
- F_λ and $F_.$ are exact and send projective modules to projective modules,
- $F_\lambda\mathcal{E}\simeq\bigoplus_{g\in G}\mathcal{B}$ and $F_.\mathcal{B}\simeq\mathcal{E}$, where $\mathcal{E}$ (or $\mathcal{B}$) is the $\mathcal{E}$-module $x\mapsto\bigoplus_{y\in\mathrm{Ob}(\mathcal{E})}{}_y\mathcal{E}_x$ (or the $\mathcal{B}$-module $x\mapsto\bigoplus_{y\in\mathrm{Ob}(\mathcal{B})}{}_y\mathcal{B}_x$, respectively),
- $F_.F_\lambda=\bigoplus_{g\in G}{}^g\mathrm{Id}_{\mathrm{MOD}(\mathcal{E})}$,
- if $X\in\mathrm{MOD}(\mathcal{B})$ verifies $X\simeq F_\lambda Y$ for some $Y\in\mathrm{MOD}(\mathcal{E})$, then $F_\lambda F_.X\simeq\bigoplus_{g\in G}X$,
- $F_\lambda(\mathrm{mod}(\mathcal{E}))\subseteq\mathrm{mod}(\mathcal{B})$, $F_\lambda(\mathrm{Mod}(\mathcal{E}))\subseteq\mathrm{Mod}(\mathcal{B})$, $F_.(\mathrm{Mod}(\mathcal{B}))\subseteq\mathrm{Mod}(\mathcal{E})$,
- $D\circ F_. = F_.\circ D$ and $D\circ F_{\lambda|\mathrm{mod}(\mathcal{E})}\simeq F_\lambda\circ D_{|\mathrm{mod}(\mathcal{E})}$ where $D=\mathrm{Hom}_k(?,k)$ is the usual duality,
- F_λ is left adjoint to $F_.$,
- $D\circ F_\lambda\circ D$ is right adjoint to $F_.$ (in particular, there is a functorial isomorphism $\mathrm{Hom}_{\mathcal{E}}(F_.M,N)\simeq\mathrm{Hom}_{\mathcal{B}}(M,F_\lambda N)$ for any $M\in\mathrm{MOD}(\mathcal{B})$ and any $N\in\mathrm{mod}(\mathcal{E})$),
- for any $M,N\in\mathrm{MOD}(\mathcal{E})$, the following mappings induced by F_λ are bijective:

$$\bigoplus_{g\in G}\mathrm{Hom}_{\mathcal{E}}\big({}^gM,N\big)\to\mathrm{Hom}_{\mathcal{B}}(F_\lambda M,F_\lambda N)\quad\text{and}$$

$$\bigoplus_{g\in G}\mathrm{Hom}_{\mathcal{E}}\big(M,{}^gN\big)\to\mathrm{Hom}_{\mathcal{B}}(F_\lambda M,F_\lambda N).$$

These properties give the following result which will be used many times in this text:

Lemma 1.2. *If $M\in\mathrm{MOD}(\mathcal{E})$ (or $M\in\mathrm{MOD}(\mathcal{B})$) has finite projective dimension, then so does $F_\lambda M$ (or $F_.(M)$, respectively).*

Let $M\in\mathrm{MOD}(\mathcal{E})$, $N\in\mathrm{MOD}(\mathcal{B})$ and $j\geqslant 1$. There is an isomorphism of vector spaces:

$$\mathrm{Ext}^j_{\mathcal{B}}(F_\lambda M,N)\simeq\mathrm{Ext}^j_{\mathcal{E}}(M,F_.N).$$

Moreover, if $M\in\mathrm{mod}(\mathcal{E})$ then there is an isomorphism of vector spaces:

$$\mathrm{Ext}^j_{\mathcal{E}}(F_.N,M)\simeq\mathrm{Ext}^j_{\mathcal{B}}(N,F_\lambda N).$$

Proof. We write $\mathcal{D}(\mathrm{MOD}(\mathcal{B}))$ and $\mathcal{D}(\mathrm{MOD}(\mathcal{E}))$ for the derived category of complexes of $\mathcal{B}$-modules and of $\mathcal{E}$-modules respectively. The first assertion is due to the fact that $F_.$ and F_λ are exact and send projective modules to projective modules. For the same reasons, $F.$ and F_λ induce $F.:\mathcal{D}(\mathrm{MOD}(\mathcal{B})) \to \mathcal{D}(\mathrm{MOD}(\mathcal{E}))$ and $F_\lambda:\mathcal{D}(\mathrm{MOD}(\mathcal{E})) \to \mathcal{D}(\mathrm{MOD}(\mathcal{B}))$ respectively and the adjunctions $(F_\lambda, F.)$ and $(F., F_\lambda)$ at the level of module categories give rise to adjunctions at the level of derived categories. Since $\mathrm{Ext}^j_{\mathcal{E}}(X,Y) = \mathrm{Hom}_{\mathcal{D}(\mathrm{MOD}(\mathcal{E}))}(Y, X[j])$ we get the announced isomorphisms. $\square$

Remark that an isomorphism of k-categories is a particular case of a Galois covering. When F is an isomorphism, $F.$ and F_λ have additional properties as shows the following lemma whose proof is a direct consequence of the definition of the push-down and pull-up functors.

Lemma 1.3. *Assume that $F:\mathcal{E} \to \mathcal{B}$ is an isomorphism of k-categories. Then $F.F_\lambda = \mathrm{Id}_{\mathrm{MOD}(\mathcal{E})}$ and $F_\lambda F. = \mathrm{Id}_{\mathrm{MOD}(\mathcal{B})}$.*

Modules of the first kind

Let $F:\mathcal{E} \to \mathcal{B}$ be a Galois covering with group G. A $\mathcal{B}$-module M is called *of the first kind with respect to F* if and only if for any indecomposable direct summand N of M there exists $\widehat{N} \in \mathrm{MOD}(\mathcal{E})$ such that $N \simeq F_\lambda \widehat{N}$. We denote by $\mathrm{ind}_1(\mathcal{B})$ (or by $\mathrm{mod}_1(\mathcal{B})$) the full subcategory of $\mathrm{ind}(\mathcal{B})$ (or of $\mathrm{mod}(\mathcal{B})$, respectively) of modules of the first kind with respect to F. Notice the following properties of $\mathrm{ind}_1(\mathcal{B})$:

- if $M \in \mathrm{ind}_1(\mathcal{B})$ and $N \in \mathrm{MOD}(\mathcal{E})$ verify $M \simeq F_\lambda N$, then $N \in \mathrm{ind}(\mathcal{E})$,
- if $M \in \mathrm{ind}_1(\mathcal{B})$ and $N, N' \in \mathrm{MOD}(\mathcal{E})$ verify $M \simeq F_\lambda N \simeq F_\lambda N'$, then there exists $g \in G$ such that $N' \simeq {}^g N$.

If $\mathcal{B}$ is connected and if $\mathcal{E} = \coprod_{i \in I} \mathcal{E}_i$, where the $\mathcal{E}_i$'s are the connected components of $\mathcal{E}$, then an indecomposable $\mathcal{B}$-module M is of the first kind with respect to F if and only if it is of the first kind with respect to $F_i:\mathcal{E}_i \hookrightarrow \mathcal{E} \to \mathcal{B}$ for any $i \in I$. More precisely, we have the following well-know lemma where we keep the established notations.

Lemma 1.4. *Let $M \in \mathrm{ind}(\mathcal{B})$. If $\widehat{M} \in \mathrm{ind}(\mathcal{E})$ is such that $F_\lambda \widehat{M} \simeq M$, then there is a unique $i \in I$ such that $\widehat{M} \in \mathrm{ind}(\mathcal{E}_i)$. In such a case, we have $M \simeq (F_i)_\lambda \widehat{M}$. Moreover, if $j \in I$ then there exists $g \in G$ such that $g(\mathcal{E}_i) = \mathcal{E}_j$, and for any such g we have: ${}^g\widehat{M} \in \mathrm{ind}(\mathcal{E}_j)$ and $(F_j)_\lambda {}^g\widehat{M} \simeq M$.*

Throughout this text A will denote a basic and connected finite dimensional k-algebra and n will denote the rank of its Grothendieck group $K_0(A)$.

2. Galois coverings associated with modules of the first kind

Throughout this section we use the following data:

- $F:\mathcal{C} \to A$ a Galois covering with group G,
- $T = T_1 \oplus \cdots \oplus T_n \in \mathrm{mod}(A)$ (with $T_i \in \mathrm{ind}(A)$) a basic tilting A-module of the first kind with respect to F,
- $\lambda_i : F_\lambda(\widehat{T_i}) \to T_i$ an isomorphism with $\widehat{T_i} \in \mathrm{ind}(\mathcal{C})$, for every $i \in \{1, \dots, n\}$.

Let $B = \mathrm{End}_A(T)$. With these data, we wish to:

(1) construct a Galois covering $F_{\widehat{T}_i,\lambda_i}$ with group G of B,
(2) study the dependence of the equivalence class of $F_{\widehat{T}_i,\lambda_i}$ on the data $\widehat{T}_i$, λ_i and on the choice of F in its equivalence class $[F]$,
(3) repeat the construction made at the first step starting from T (viewed as a basic tilting B-module) and the Galois covering $F_{\widehat{T}_i,\lambda_i}$. This will give a Galois covering of $\mathrm{End}_B(T)$ which will be compared with F using the isomorphism $\rho_A : A \xrightarrow{\sim} \mathrm{End}_B(T)$.

2.1. Construction of the Galois covering $F_{\widehat{T}_i,\lambda_i}$

Let $\mathcal{E}nd_{\mathcal{C}}(\bigoplus_{g,i} {}^g\widehat{T}_i)$ be the full subcategory of $\mathrm{mod}(\mathcal{C})$ with objects the modules ${}^g\widehat{T}_i$ for $g \in G$ and $i \in \{1,\dots,n\}$. In particular, if $g \neq g'$, then we consider ${}^g\widehat{T}_i$ and ${}^{g'}\widehat{T}_i$ as different objects of this category although they may be isomorphic as $\mathcal{C}$-modules.

Remark 2.1.

1. The $\mathcal{C}$-modules $\bigoplus_{g,i} {}^g\widehat{T}_i$ and $F.T$ are isomorphic.
2. If G is a finite group, then $\mathcal{C}$ is a finite dimensional k-algebra. In particular, $\mathcal{E}nd_{\mathcal{C}}(\bigoplus_{g,i} {}^g\widehat{T}_i)$ and $\mathrm{End}_{\mathcal{C}}(F.T)$ are isomorphic k-algebras.
3. The G-action on $\mathrm{mod}(\mathcal{C})$ naturally endows $\mathcal{E}nd_{\mathcal{C}}(\bigoplus_{g,i} {}^g\widehat{T}_i)$ with a structure of free G-category.
4. $\mathcal{E}nd_{\mathcal{C}}(\bigoplus_{i,g} {}^g\widehat{T}_i)$ is locally bounded if and only if $G_{\widehat{T}_i} = 1$ for any i. This is equivalent to say that $F.T$ is a basic $\mathcal{C}$-module.

The isomorphisms $\lambda_1,\dots,\lambda_n$ define the following functor:

$$\begin{aligned} F_{\widehat{T}_i,\lambda_i} : \mathcal{E}nd_{\mathcal{C}}\Big(\bigoplus_{g,i} {}^g\widehat{T}_i\Big) &\to B, \\ {}^g\widehat{T}_i &\mapsto T_i, \\ {}^g\widehat{T}_i \xrightarrow{u} {}^h\widehat{T}_j &\mapsto T_i \xrightarrow{\lambda_j F_\lambda(u)\lambda_i^{-1}} T_j. \end{aligned}$$

Lemma 2.2. *The functor $F_{\widehat{T}_i,\lambda_i} : \mathcal{E}nd_{\mathcal{C}}(\bigoplus_{i,g} {}^g\widehat{T}_i) \to B$ is a Galois covering with group G.*

Proof. For simplicity, we shall write $\mathcal{C}'$ for $\mathcal{E}nd_{\mathcal{C}}(\bigoplus_{i,g} {}^g\widehat{T}_i)$ and $F' : \mathcal{C}' \to B$ for $F_{\widehat{T}_i,\lambda_i}$. Recall (see Remark 2.1) that G acts freely on $\mathcal{C}'$. Moreover, we have $F' \circ g = F'$ by construction of F'. So, F' defines a commutative diagram of k-categories and k-linear functors:

$$\begin{tikzcd} \mathcal{C}' \arrow[d] \arrow[dr, "F'"] & \\ \mathcal{C}'/G \arrow[r, "\overline{F'}"'] & B, \end{tikzcd} \qquad (\star)$$

where $\mathcal{C}' \to \mathcal{C}'/G$ is the quotient functor. From the properties verified by F_λ (see Section 1) we infer that F' is a covering functor. Since $\mathcal{C}' \to \mathcal{C}'/G$ is also a covering functor we deduce that so is $\overline{F'}$ (see Lemma 1.1). Finally, $\overline{F'}$ restricts to a bijective mapping $\mathrm{Ob}(\mathcal{C}')/G = \{{}^g\widehat{T}_i \mid g \in G,$

$i \in \{1, \dots, n\}\}/G \to \mathrm{Ob}(B) = \{T_1, \dots, T_n\}$ so $\overline{F'}$ is an isomorphism. Thus, F' is a Galois covering with group G. $\square$

Since $F_{\widehat{T}_i,\lambda_i}$ is a Galois covering, it is natural to ask whether $\mathcal{E}nd_{\mathcal{C}}(\bigoplus_{i,g} {}^g\widehat{T}_i)$ is connected or not. The following lemma partially answers this question.

Lemma 2.3. *If $\mathcal{C}$ is not connected, then $\mathcal{E}nd_{\mathcal{C}}(\bigoplus_{i,g} {}^g\widehat{T}_i)$ is not connected.*

Proof. For simplicity let us write $\mathcal{C}'$ for $\mathcal{E}nd_{\mathcal{C}}(\bigoplus_{i,g} {}^g\widehat{T}_i)$. Assume that $\mathcal{C}$ is not connected and let $\mathcal{C} = \coprod_{x\in I} \mathcal{C}_x$ where the $\mathcal{C}_x$'s are the connected components of $\mathcal{C}$. For $i \in \{1, \dots, n\}$, we have $\widehat{T}_i \in \mathrm{ind}(\mathcal{C})$, so there exists a unique $x_i \in I$ such that $\widehat{T}_i \in \mathrm{ind}(\mathcal{C}_{x_i})$. Let us set:

$$G_{x_1} = \left\{g \in G \;\middle|\; g(\mathcal{C}_{x_1}) = \mathcal{C}_{x_1}\right\}.$$

Let $i \in \{1, \dots, n\}$, since G acts transitively on $\{\mathcal{C}_x \mid x \in I\}$, there exists $g_i \in G$ such that $g_i(\mathcal{C}_{x_1}) = \mathcal{C}_{x_i}$ (in particular $g_1 \in G_{x_1}$). Therefore, ${}^{g_i^{-1}}\widehat{T}_i \in \mathrm{mod}(\mathcal{C}_{x_1})$ for every $i \in \{1, \dots, n\}$. Let us set O to be the following set of objects of $\mathcal{C}'$:

$$O := \left\{{}^g\widehat{T}_i \;\middle|\; i \in \{1, \dots, n\} \text{ and } gg_i \in G_{x_1}\right\} \subseteq \mathrm{Ob}(\mathcal{C}').$$

Remark that O satisfies the following:

- $O \neq \emptyset$ because $\widehat{T}_1 \in O$.
- Since $\mathcal{C}$ is not connected and since G acts transitively on $\{\mathcal{C}_x \mid x \in I\}$ we have $G_{x_1} \subsetneq G$. Let $g \in G \setminus G_{x_1}$, then $gg_1 \notin G_{x_1}$ and ${}^g\widehat{T}_1 \notin O$. Hence $O \subsetneq \mathrm{Ob}(\mathcal{C})$.
- For any ${}^g\widehat{T}_i \in \mathrm{Ob}(\mathcal{C}')$, we have ${}^g\widehat{T}_i \in O$ if and only if ${}^g\widehat{T}_i \in \mathrm{ind}(\mathcal{C}_{x_1})$. As a consequence, there is no non-zero morphism in $\mathcal{C}'$ between an object in O and an object in $\mathrm{Ob}(\mathcal{C}') \setminus O$.

As a consequence, $\mathcal{C}'$ is not connected. $\square$

2.2. *Independence of the equivalence class of $F_{\widehat{T}_i,\lambda_i}$ on the data $F, \widehat{T}_i, \lambda_i$*

In the two following lemmata, we examine the dependence of the equivalence class $[F_{\widehat{T}_i,\lambda_i}]$ of $F_{\widehat{T}_i,\lambda_i}$ on the choice of $\widehat{T}_1, \dots, \widehat{T}_n, \lambda_1, \dots, \lambda_n$ and on the choice of F in its equivalence class $[F]$.

Lemma 2.4. *For each $i \in \{1, \dots, n\}$, let $\mu_i : F_\lambda \overline{T}_i \to T_i$ be an isomorphism with $\overline{T}_i \in \mathrm{ind}(\mathcal{C})$. Then $F_{\widehat{T}_i,\lambda_i}$ and $F_{\overline{T}_i,\mu_i}$ are equivalent.*

Proof. We need to exhibit a commutative square:

$$\begin{tikzcd} \mathcal{E}nd_{\mathcal{C}}(\bigoplus_{i,g} {}^g\overline{T}_i) \arrow[r, "\varphi"] \arrow[d, "F_{\overline{T}_i,\mu_i}"'] & \mathcal{E}nd_{\mathcal{C}}(\bigoplus_{i,g} {}^g\widehat{T}_i) \arrow[d, "F_{\widehat{T}_i,\lambda_i}"] \\ B \arrow[r, "\psi"] & B, \end{tikzcd} \qquad (\star)$$

where φ, ψ are isomorphisms and where $\psi(x)=x$ for any $x\in \mathrm{Ob}(B)=\{T_1,\dots,T_n\}$. Let $i\in\{1,\dots,n\}$. We have $F_\lambda\overline{T}_i\simeq T_i\simeq F_\lambda\widehat{T_i}$, so there exists an isomorphism $\theta_i:\overline{T}_i\xrightarrow{\sim}{}^{g_i}\widehat{T_i}$ with $g_i\in G$. Let us define φ by:

$$\varphi:\mathcal{E}nd_{\mathcal{C}}\Big(\bigoplus_{i,g}{}^{g}\overline{T}_i\Big)\to\mathcal{E}nd_{\mathcal{C}}\Big(\bigoplus_{i,g}{}^{g}\widehat{T_i}\Big),$$

$${}^{g}\overline{T}_i\mapsto{}^{gg_i}\widehat{T_i},$$

$${}^{g}\overline{T}_i\xrightarrow{u}{}^{h}\overline{T}_j\mapsto{}^{gg_i}\widehat{T_i}\xrightarrow{{}^{h}\theta_j u\,{}^{g}\theta_i^{-1}}{}^{hg_j}\widehat{T}_j.$$

Then φ is an isomorphism of k-categories. Notice that θ_i defines an isomorphism $F_\lambda\theta_i:F_\lambda\overline{T}_i\to F_\lambda\widehat{T_i}$. So we can define ψ by:

$$\psi:B\to B,$$

$$T_i\mapsto T_i,$$

$$T_i\xrightarrow{u}T_j\mapsto\psi(u),$$

where $\psi(u)$ is the composition:

$$T_i\xrightarrow{\lambda_i^{-1}}F_\lambda\widehat{T_i}\xrightarrow{F_\lambda\theta_i^{-1}}F_\lambda\overline{T}_i\xrightarrow{\mu_i}T_i\xrightarrow{u}T_j\xrightarrow{\mu_j^{-1}}F_\lambda\overline{T}_j\xrightarrow{F_\lambda\theta_j}F_\lambda\widehat{T_j}\xrightarrow{\lambda_j}T_j.$$

So ψ is an isomorphism of k-categories which restricts to the identity map on $\mathrm{Ob}(B)$. Moreover φ and ψ make $(\star)$ commutative. $\square$

In the following lemma, we show that, under additional hypotheses on T, the equivalence class of $F_{\widehat{T_i},\lambda_i}$ does not depend on the choice of F in $[F]$.

Lemma 2.5. *Assume that $F':\mathcal{C}'\to A$ is a Galois covering (with group G) equivalent to F and assume that T verifies the following condition:*

$(H_{A,T})$ *"$\psi.T_i\simeq T_i$ for any i and for any isomorphism $\psi:A\xrightarrow{\sim}A$ which restricts to the identity map on* $\mathrm{Ob}(A)$."

Then T is of the first kind with respect to F'. For each $i\in\{1,\dots,n\}$ let $\mu_i:F'_\lambda\overline{T}_i\to T_i$ be an isomorphism with $\overline{T}_i\in\mathrm{ind}(\mathcal{C}')$. Then $F'_{\overline{T}_i,\mu_i}$ and $F_{\widehat{T_i},\lambda_i}$ are equivalent.

Proof. Let us fix an isomorphism between F and F':

$$\begin{tikzcd}\mathcal{C}' \arrow[r, "\varphi"] \arrow[d, "F'"'] & \mathcal{C} \arrow[d, "F"] \\ A \arrow[r, "\psi"] & A.\end{tikzcd}$$

Let us set $\nu : \mathrm{Aut}(\mathcal{C}') \to \mathrm{Aut}(\mathcal{C})$ to be the isomorphism of groups (recall that $\mathrm{Aut}(\mathcal{C}') = G$ and $\mathrm{Aut}(\mathcal{C}) = G$):

$$\begin{aligned}\nu : \mathrm{Aut}(\mathcal{C}') &\to \mathrm{Aut}(\mathcal{C}),\\ g &\mapsto \varphi \circ g \circ \varphi^{-1}.\end{aligned}$$

Recall that any $g \in \mathrm{Aut}(\mathcal{C}) = G$ (or any $g \in \mathrm{Aut}(\mathcal{C}') = G$) defines an automorphism g of $\mathrm{MOD}(\mathcal{C})$ (or of $\mathrm{MOD}(\mathcal{C}')$, respectively). Therefore, we have an equality of functors $\mathrm{MOD}(\mathcal{C}') \to \mathrm{MOD}(A)$, for every $g \in \mathrm{Aut}(\mathcal{C}')$:

$$\varphi_\lambda \circ g = \nu(g) \circ \varphi_\lambda.$$

Let us fix an isomorphism $\theta_i : \psi.T_i \to T_i$, for each i, and let us set $\overline{T}_i = \varphi.\widehat{T_i}$. In particular: $\varphi_\lambda \overline{T}_i = \widehat{T_i}$ (see Lemma 1.3). Since $\psi.\psi_\lambda = \mathrm{Id}_{\mathrm{MOD}(A)}$ (see Lemma 1.3) and $\psi F' = F\varphi$, we infer that:

$$F'_\lambda \overline{T}_i = \psi.\psi_\lambda F'_\lambda \overline{T_i} = \psi.F_\lambda \varphi_\lambda \overline{T}_i = \psi.F_\lambda \widehat{T}_i.$$

Therefore, we get for each i an isomorphism $\mu_i : F'_\lambda \overline{T}_i \to T_i$ equal to the composition:

$$\mu_i : F'_\lambda \overline{T}_i = \psi.F_\lambda \widehat{T_i} \xrightarrow{\psi.\lambda_i} \psi.T_i \xrightarrow{\theta_i} T_i.$$

This proves that T is of the first kind with respect to F'. According to the preceding subsection, this defines the Galois covering with group G:

$$F'_{\overline{T}_i,\mu_i} : \mathcal{E}nd_{\mathcal{C}'}\left(\bigoplus_{g,i} {}^g\overline{T}_i\right) \to B.$$

Thanks to Lemma 2.4 we only need to prove that $F'_{\overline{T}_i,\mu_i}$ and $F_{\widehat{T}_i,\lambda_i}$ are equivalent.

Firstly, we have the following functor induced by φ_λ:

$$\begin{aligned}\overline{\varphi} : \mathcal{E}nd_{\mathcal{C}'}\left(\bigoplus_{i,g} {}^g\overline{T}_i\right) &\to \mathcal{E}nd_{\mathcal{C}}\left(\bigoplus_{i,g} {}^g\widehat{T_i}\right),\\ {}^g\overline{T}_i &\mapsto {}^{\nu(g)}\widehat{T_i} = \varphi_\lambda\, {}^g\overline{T}_i,\\ {}^g\overline{T}_i \xrightarrow{u} {}^h\overline{T}_j &\mapsto {}^{\nu(g)}\widehat{T_i} \xrightarrow{\varphi_\lambda u} {}^{\nu(h)}\widehat{T}_j.\end{aligned}$$

Since $\nu : G \to G$ is an isomorphism and because of the equalities $\varphi_\lambda \varphi. = \mathrm{Id}_{\mathrm{MOD}(\mathcal{C})}$ and $\varphi.\varphi_\lambda = \mathrm{Id}_{\mathrm{MOD}(\mathcal{C})}$ (see Lemma 1.3), the functor $\overline{\varphi}$ is an isomorphism.

Secondly, we have the following functor induced by ψ_λ:

$$\begin{aligned}\overline{\psi} : B &\to B,\\ T_i &\mapsto T_i,\\ T_i \xrightarrow{u} T_j &\mapsto T_i \xrightarrow{\psi_\lambda(\theta_j^{-1} u \theta_i)} T_j.\end{aligned}$$

Since $\psi_\lambda\psi_. = \psi_.\psi_\lambda = \mathrm{Id}_{\mathrm{MOD}(A)}$, the functor $\overline{\psi}$ is a well-defined isomorphism and restricts to the identity map on $\mathrm{Ob}(B)$. Therefore, we have a diagram whose horizontal arrows are isomorphisms and whose bottom horizontal arrow restricts to the identity map on the set of objects:

$$\begin{tikzcd}
\mathcal{E}nd_{\mathcal{C}'}(\bigoplus_{i,g} {}^g\overline{T}_i) \arrow[r, "\overline{\varphi}"] \arrow[d, "F'_{\overline{T}_i,\mu_i}"'] & \mathcal{E}nd_{\mathcal{C}}(\bigoplus_{i,g} {}^g\widehat{T_i}) \arrow[d, "F_{\widehat{T_i},\lambda_i}"] \\
B \arrow[r, "\overline{\psi}"] & B.
\end{tikzcd}$$

This diagram is commutative. Indeed, for any ${}^g\overline{T}_i \xrightarrow{u} {}^h\overline{T}_j$ we have:

$$\begin{aligned}
\overline{\psi} F'_{\overline{T}_i,\mu_i}(u) &= \overline{\psi}\big(\mu_j F'_\lambda(u)\mu_i^{-1}\big) = \psi_\lambda\big(\theta_j^{-1}\mu_j F'_\lambda(u)\mu_i^{-1}\theta_i\big) \\
&= \psi_\lambda\big(\theta_j^{-1}\theta_j\psi_.(\lambda_j)F'_\lambda(u)\psi_.(\lambda_i)^{-1}\theta_i^{-1}\theta_i\big) \\
&= \lambda_j\big(\psi_\lambda F'_\lambda\big)(u)\lambda_i^{-1} \quad \text{because } \psi_\lambda\psi_. = \mathrm{Id}_{\mathrm{MOD}(\mathcal{C})} \\
&= \lambda_j(F_\lambda\varphi_\lambda)(u)\lambda_i^{-1} \quad \text{because } F\varphi = \psi F' \\
&= F_{\widehat{T_i},\lambda_i}\big(\varphi_\lambda(u)\big) = F_{\widehat{T_i},\lambda_i}\overline{\varphi}(u).
\end{aligned}$$

This proves that $F'_{\overline{T}_i,\mu_i}$ and $F_{\widehat{T_i},\lambda_i}$ are equivalent. □

Later, we shall prove that if T is a basic tilting A-module lying in the connected component of $\vec{\mathcal{K}}_A$ containing A, then the hypothesis $(H_{A,T})$ in the preceding lemma is automatically verified. As a consequence, for these tilting A-modules, the property "to be of the first kind with respect to an equivalence class of Galois coverings of A" does make sense. However, the hypothesis $(H_{A,T})$ is not verified for any A-module T as the following example shows.

Example 2.6. Let A be the path algebra of the following quiver:

$$\begin{tikzcd}
& 2 \arrow[dr, "c"] & \\
1 \arrow[ur, "b"] \arrow[rr, "a"'] & & 3.
\end{tikzcd}$$

Here $n = 3$ and we have an isomorphism of k-categories: $\psi : A \xrightarrow{\sim} A$ such that $\psi(x) = x$ for any $x \in \mathrm{Ob}(A)$, $\psi(a) = a + cb$, $\psi(b) = b$ and $\psi(c) = c$. For any integer i, let T_i be the following A-module:

$$\begin{tikzcd}
& k \arrow[dr, "1"] & \\
k \arrow[ur, "1"] \arrow[rr, "i"'] & & k.
\end{tikzcd}$$

Then:

- T_i and T_{i+1} are not isomorphic, for any i,
- if $char(k) \neq 2$, then T_1, T_2, T_3 are pairwise non-isomorphic,
- $\psi.T_i = T_{i+1}$ for any i.

In particular, if $char(k) \neq 2$, then hypothesis $(H_{A,T})$ is not satisfied for $T = T_1 \oplus T_2 \oplus T_3$. Remark that T is not tilting. Indeed, for any i, we have $\mathrm{Ext}^1_A(T_i, T_i) \simeq k$ because $\tau_A(T_i) \simeq T_i$.

Remark 2.1, Lemmas 2.4 and 2.5 imply the following result:

Proposition 2.7. *Assume that the hypothesis* $(H_{A,T})$ *is satisfied* (*see Lemma* 2.5)*. The equivalence class* $[F]$ *of* F *and the basic tilting* A*-module* $T = T_1 \oplus \cdots \oplus T_n$ *of the first kind with respect to* $[F]$ (*with* $T_i \in \mathrm{ind}(A)$) *uniquely define an equivalence class of Galois coverings of* B *with group* G *and which admits* $F_{\widehat{T_i},\lambda_i}$ *as a representative.*

With the notations and hypotheses of the previous proposition, the equivalence class of the Galois covering $F_{\widehat{T_i},\lambda_i}$ is denoted by $[F]_T : \mathcal{E}nd_{\mathcal{C}}(F.T) \to B$.

2.3. *Comparison of* $[F]$ *and* $([F]_T)_T$

For short, let us write $\mathcal{C}'$ for $\mathcal{E}nd_{\mathcal{C}}(\bigoplus_{g,i} {}^g\widehat{T_i})$ and F' for $F_{\widehat{T_i},\lambda_i}$. In this subsection we shall not assume that the hypothesis $(H_{A,T})$ of Lemma 2.5 is satisfied, except for the last proposition. Starting from F and from the isomorphisms $\lambda_i : F_\lambda \widehat{T_i} \xrightarrow{\sim} T_i$, $i \in \{1, \ldots, n\}$, we have constructed the Galois covering F' of B. One may try to perform the same construction starting from F' in order to get a Galois covering F'' of $\mathrm{End}_B(T) \simeq A$ and eventually compare F'' with F. For this purpose, we need to prove that T is of the first kind with respect to F'. Let us fix a lifting $L : \mathrm{Ob}(A) \to \mathrm{Ob}(\mathcal{C})$ of the surjective mapping $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(A)$. For $x \in \mathrm{Ob}(A)$, let $\widehat{T(x)}$ be the $\mathcal{C}'$-module such that:

- $\widehat{T(x)}({}^g\widehat{T_i}) = \widehat{T_i}(g^{-1}L(x))$ for any ${}^g\widehat{T_i} \in \mathrm{Ob}(\mathcal{C}')$,
- $\widehat{T(x)}({}^g\widehat{T_i} \xrightarrow{u} {}^h\widehat{T_j})$ is equal to $\widehat{T_i}(g^{-1}L(x)) \xrightarrow{u_{L(x)}} \widehat{T_j}(h^{-1}L(x))$ for any $u \in {}_{{}^h\widehat{T_j}}\mathcal{C}'_{{}^g\widehat{T_i}}$.

Therefore, for any $i \in \{1, \ldots, n\}$, we have:

$$\big(F'_\lambda \widehat{T(x)}\big)(T_i) = \bigoplus_{g \in G} \widehat{T(x)}\big({}^g\widehat{T_i}\big) = \bigoplus_{g \in G} \widehat{T_i}\big(g^{-1}L(x)\big) = (F_\lambda \widehat{T_i})(x).$$

So we may set $(\mu_x)_{T_i} : (F'_\lambda \widehat{T(x)})(T_i) \to (T(x))(T_i)$ to be equal to $(F_\lambda \widehat{T_i})(x) \xrightarrow{(\lambda_i)_x} T_i(x)$.

Lemma 2.8. *The linear isomorphisms* $(\mu_x)_{T_i}$ ($i \in \{1, \ldots, n\}$) *define an isomorphism of* B*-modules*:

$$\mu_x : F'_\lambda \widehat{T(x)} \xrightarrow{\sim} T(x).$$

Proof. We only need to prove that μ_x is a morphism of B-modules. Let $u \in {}_{{}^t\widehat{T}_j}\mathcal{C}'_{{}^s\widehat{T}_i}$ so that $F'(u) \in {}_{T_j}B_{T_i}$, and let us prove that the following diagram commutes:

$$\begin{tikzcd} (F'_\lambda\widehat{T(x)})(T_i) \arrow[rr, "(\mu_x)_{T_i}=(\lambda_i)_x"] \arrow[d, "(F'_\lambda\widehat{T(x)})(F'(u))"'] & & T_i(x) \arrow[d, "(T(x))(F'(u))=F'(u)_x"] \\ (F'_\lambda\widehat{T(x)})(T_j) \arrow[rr, "(\mu_x)_{T_j}=(\lambda_j)_x"] & & T_j(x). \end{tikzcd} \tag{$\star$}$$

Let $g \in G$ and let us compute the restriction of $F'(u)_x \circ (\lambda_i)_x$ to $\widehat{T(x)}({}^g\widehat{T}_i)$. Recall that $F'(u)_x$ is equal to the composition:

$$T_i(x) \xrightarrow{(\lambda_i^{-1})_x} \big(F_\lambda {}^s\widehat{T}_i\big)(x) \xrightarrow{(F_\lambda u)_x} \big(F_\lambda {}^t\widehat{T}_j\big)(x) \xrightarrow{(\lambda_j)_x} T_j(x).$$

Moreover, the restriction of $(F_\lambda u)_x$ to $\widehat{T(x)}({}^g\widehat{T}_i) = \widehat{T}_i(g^{-1}L(x))$ is (by construction of the push-down functor) equal to $\widehat{T}_i(g^{-1}L(x)) \xrightarrow{u_{sg^{-1}L(x)}} \widehat{T}_j(t^{-1}sg^{-1}L(x))$. Thus, the restriction of $F'(u)_x \circ (\lambda_i)_x$ to $\widehat{T(x)}({}^g\widehat{T}_i)$ is equal to the composition:

$$\widehat{T}_i\big(g^{-1}L(x)\big) \xrightarrow{u_{sg^{-1}L(x)}} \widehat{T}_j\big(t^{-1}sg^{-1}L(x)\big) \xrightarrow{(\lambda_j)_x} T_j(x). \tag{i}$$

On the other hand, the restriction of $(F'_\lambda\widehat{T(x)})(F'(u))$ to $\widehat{T(x)}({}^g\widehat{T}_i) = \widehat{T(x)}({}^{gs^{-1}s}\widehat{T}_i)$ is (by construction of the push-down functor) equal to:

$$\widehat{T(x)}\big({}^g\widehat{T}_i\big) \xrightarrow{\widehat{T(x)}({}^{gs^{-1}}u)} \widehat{T(x)}\big({}^{gs^{-1}t}\widehat{T}_j\big), \tag{ii}$$

and $\widehat{T(x)}({}^{gs^{-1}}u) = ({}^{gs^{-1}}u)_{L(x)} = u_{sg^{-1}L(x)}$. This last equality, together with (i) and (ii), proves that the diagram $(\star)$ commutes. $\square$

Thanks to Lemma 2.8, we have a Galois covering $F'_{\widehat{T(x)},\mu_x} : \mathcal{E}nd_{\mathcal{C}'}(\bigoplus_{g,x} {}^g\widehat{T(x)}) \to \mathrm{End}_B(T)$ with group G. For short, we shall write $\mathcal{C}''$ for $\mathcal{E}nd_{\mathcal{C}'}(\bigoplus_{g,x} {}^g\widehat{T(x)})$ and F'' for $F'_{\widehat{T(x)},\mu_x}$. The following lemma relates F'' and F.

Lemma 2.9. *There exists an isomorphism of k-categories $\psi : \mathcal{C} \xrightarrow{\sim} \mathcal{C}''$ such that the following diagram is commutative*:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\psi"] \arrow[d, "F"'] & \mathcal{C}'' \arrow[d, "F''"] \\ A \arrow[r, "\rho_A"] & \mathrm{End}_B(T). \end{tikzcd}$$

In particular, F and $\rho_A^{-1}F''$ are equivalent as Galois coverings of A.

Proof. Since G acts freely on $\mathrm{Ob}(\mathcal{C})$ and since $L:\mathrm{Ob}(A)\to\mathrm{Ob}(\mathcal{C})$ lifts $F:\mathrm{Ob}(\mathcal{C})\to\mathrm{Ob}(A)$, any $x\in\mathrm{Ob}(\mathcal{C})$ is equal to $gL(x')$ with $g\in G, x'\in\mathrm{Ob}(A)$ uniquely determined by x. Let $\psi:\mathcal{C}\to\mathcal{C}''$ be as follows:

- $\psi(gL(x))={}^g\widehat{T(x)}$ for any $gL(x)\in\mathrm{Ob}(\mathcal{C})$,
- for $u\in {}_{hL(y)}\mathcal{C}_{gL(x)}$, we let $\psi(u):{}^g\widehat{T(x)}\to{}^h\widehat{T(y)}$ be the morphism of $\mathcal{C}'$-modules such that for any ${}^s\widehat{T_i}\in\mathrm{Ob}(\mathcal{C}')$, $\psi(u)_{{}^s\widehat{T_i}}$ is equal to:

$$\widehat{T_i}\big(s^{-1}u\big):\widehat{T_i}\big(s^{-1}gL(x)\big)\to\widehat{T_i}\big(s^{-1}hL(y)\big).$$

Let us prove the following facts:

1. $\psi(u)$ is a morphism of $\mathcal{C}'$-modules for any $u\in {}_{hL(y)}\mathcal{C}_{gL(x)}$,
2. ψ is a functor,
3. $F''\circ\psi=\rho_A\circ F$,
4. ψ is an isomorphism.

(1) Let $u\in {}_{hL(y)}\mathcal{C}_{gL(x)}$. We need to prove that for any $f\in {}_{{}^t\widehat{T_j}}\mathcal{C}'_{{}^s\widehat{T_i}}$, the following diagram commutes:

$$\begin{tikzcd}
{}^g\widehat{T(x)}({}^s\widehat{T_i}) \arrow[r, "\psi(u)_{{}^s\widehat{T_i}}"] \arrow[d, "{}^g\widehat{T(x)}(f)"'] & {}^h\widehat{T(y)}({}^s\widehat{T_i}) \arrow[d, "{}^h\widehat{T(y)}(f)"] \\
{}^g\widehat{T(x)}({}^t\widehat{T_j}) \arrow[r, "\psi(u)_{{}^t\widehat{T_j}}"] & {}^h\widehat{T(y)}({}^t\widehat{T_j}).
\end{tikzcd}$$

By construction, this diagram is equal to:

$$\begin{tikzcd}
\widehat{T_i}(s^{-1}gL(x)) \arrow[r, "\widehat{T_i}(s^{-1}u)"] \arrow[d, "f_{gL(x)}"'] & \widehat{T_i}(s^{-1}hL(y)) \arrow[d, "f_{hL(y)}"] \\
\widehat{T_j}(t^{-1}gL(x)) \arrow[r, "\widehat{T_j}(t^{-1}u)"] & \widehat{T_j}(t^{-1}hL(y))
\end{tikzcd}$$

and the latter is commutative because $f:{}^s\widehat{T_i}\to{}^t\widehat{T_j}$ is a morphism of $\mathcal{C}$-modules. This proves that $\psi(u)$ is a morphism of $\mathcal{C}'$-modules for any morphism u in $\mathcal{C}$.

(2) One easily checks that $\psi(1_{gL(x)})=\mathrm{Id}_{{}^g\widehat{T(x)}}$ for any $gL(x)\in\mathrm{Ob}(\mathcal{C})$. Let u,v be morphisms in $\mathcal{C}$ such that the composition vu exists. Then, for any ${}^s\widehat{T_j}\in\mathrm{Ob}(\mathcal{C}')$:

$$\begin{aligned}\big(\psi(v)\circ\psi(u)\big)_{{}^s\widehat{T_j}}&=\psi(v)_{{}^s\widehat{T_j}}\circ\psi(u)_{{}^s\widehat{T_j}}=\widehat{T_j}\big(s^{-1}v\big)\circ\widehat{T_j}\big(s^{-1}u\big)\\&=\widehat{T_j}\big(s^{-1}(v\circ u)\big)=\psi(v\circ u)_{{}^s\widehat{T_j}}.\end{aligned}$$

So $\psi:\mathcal{C}\to\mathcal{C}''$ is a functor.

(3) Let $gL(x) \in \mathrm{Ob}(\mathcal{C})$. Then:

$$F'' \circ \psi\big(gL(x)\big) = F''\big({}^{g}\widehat{T(x)}\big) = T(x) = \rho_A(x) = \rho_A \circ F\big(gL(x)\big).$$

Let $u \in {}_{hL(y)}\mathcal{C}_{gL(x)}$ and let us prove that $F''\psi(u) = \rho_A F(u)$. Let $T_i \in \mathrm{Ob}(B)$. Then:

$$\begin{cases} (F''\psi(u))_{T_i} = T_i(x) \xrightarrow{(\lambda_i^{-1})_x} (F'_\lambda\, {}^{g}\widehat{T(x)})(T_i) \xrightarrow{(F'_\lambda(\psi(u)))_{T_i}} (F'_\lambda\, {}^{h}\widehat{T(y)})(T_i) \xrightarrow{(\lambda_i)_y} T_i(y), \\ (\rho_A F(u))_{T_i} = T_i(x) \xrightarrow{T_i(F(u))} T_i(y). \end{cases}$$

Recall that $(F'_\lambda\, {}^{g}\widehat{T(x)})(T_i) = \bigoplus_{s\in G} {}^{g}\widehat{T(x)}({}^{s}\widehat{T_i})$ and that ${}^{g}\widehat{T(x)}({}^{s}\widehat{T_i}) = \widehat{T_i}(s^{-1}gL(x))$, for any $s \in G$. Let $s \in G$. Then, the restriction of $(F'_\lambda(\psi(u)))_{T_i}$ to ${}^{g}\widehat{T(x)}({}^{s}\widehat{T_i})$ is equal to:

$$\widehat{T_i}\big(s^{-1}gL(x)\big) \xrightarrow{\psi(u)_{{}^{s}\widehat{T_i}} = \widehat{T_i}(s^{-1}u)} \widehat{T_i}\big(s^{-1}hL(y)\big).$$

Therefore, for any $s \in G$, the restriction of $(\lambda_i^{-1})_y \circ (F''\psi(u))_{T_i} \circ (\lambda_i)_x$ (or of $(\lambda_i^{-1})_y \circ (\rho_A F(u))_{T_i} \circ (\lambda_i)_x$) to ${}^{g}\widehat{T(x)}({}^{s}\widehat{T_i}) = \widehat{T_i}(s^{-1}gL(x))$ is equal to:

$$\widehat{T_i}\big(s^{-1}gL(x)\big) \xrightarrow{\widehat{T_i}(s^{-1}u)} \widehat{T_i}\big(s^{-1}hL(y)\big)$$

$$\Big(\text{or to } \widehat{T_i}\big(s^{-1}gL(x)\big) \xrightarrow{(\lambda_i^{-1})_y T_i(F(u))(\lambda_i)_x} \widehat{T_i}\big(s^{-1}hL(y)\big), \text{ respectively}\Big).$$

Since $\lambda_i : F_\lambda({}^{s}\widehat{T_i}) \to T_i$ is an isomorphism of A-modules, $(\lambda_i^{-1})_y \circ T_i(F(u)) \circ (\lambda_i)_x$ equals $\widehat{T_i}(s^{-1}u)$. We infer that $(\lambda_i^{-1})_y \circ (F''\psi(u))_{T_i} \circ (\lambda_i)_x$ and $(\lambda_i^{-1})_y \circ (\rho_A F(u))_{T_i} \circ (\lambda_i)_x$ coincide on $\widehat{T}(s^{-1}gL(x))$, for any $s \in G$. As a consequence, $(F''\psi(u))_{T_i} = (\rho_A F(u))_{T_i}$, for any $T_i \in \mathrm{Ob}(B)$. This proves that $F'' \circ \psi(u) = \rho_A \circ F(u)$ for any morphism u in $\mathcal{C}$. In other words: $F'' \circ \psi = \rho_A \circ F$.

(4) Let us prove that ψ is an isomorphism. Since F'' and $\rho_A \circ F$ are covering functors, Lemma 1.1 implies that so does ψ. Since ψ restricts to a bijective mapping on objects, we deduce that ψ is an isomorphism. □

Thanks to Lemma 2.9 we can complete Lemma 2.3 concerning the connectedness of $\mathcal{C}'$. The following proposition will be useful in the sequel, it is a direct consequence of Lemmas 2.3 and 2.9.

Proposition 2.10. *$\mathcal{C}$ is connected if and only if $\mathcal{C}' = \mathcal{E}nd_{\mathcal{C}}(\bigoplus_{g,i} \widehat{T_i})$ is connected.*

We finish this subsection with the following proposition which compares the equivalence class of F and $([F]_T)_T$ when the latter is well-defined (see Definition 2.7). It is a direct consequence of Lemma 2.9. Notice that $\rho_A^{-1} \circ ([F]_T)_T$ is an equivalence class of Galois coverings of A.

Proposition 2.11. *Assume that both conditions $(H_{A,T})$ and $(H_{B,T})$ are satisfied. Then, the equivalence class $[F]$ of F coincides with $\rho_A^{-1} \circ ([F]_T)_T$.*

3. Tilting modules of the first kind

Let $F:\mathcal{C}\to A$ be a Galois covering with group G and with $\mathcal{C}$ locally bounded. The aim of this section is to give "simple" sufficient conditions which guarantee the following facts:

- T is of the first kind with respect to F,
- $F.T$ is a basic $\mathcal{C}$-module,
- the hypothesis $(H_{A,T})$ is satisfied (see Lemma 2.5), that is, $\psi.N\simeq N$ for any direct summand N of T and for any automorphism $\psi: A\xrightarrow{\sim} A$ which restricts to the identity map on objects.

We begin with the following proposition.

Proposition 3.1. *Assume that T and T' lie in a same connected component of $\vec{\mathcal{K}}_A$. Then*:

$$T\in \mathrm{mod}_1(A) \textit{ if and only if } T'\in \mathrm{mod}_1(A).$$

In particular, if $T'=A$ or $T'=DA$, then $T\in \mathrm{mod}_1(A)$.

Proof. Since $A, DA\in \mathrm{mod}_1(A)$, we only need to prove the equivalence of the proposition under the assumption: there is an arrow $T\to T'$ in $\vec{\mathcal{K}}_A$. Let us assume that $T\in \mathrm{mod}_1(A)$. Since $T\to T'$ is an arrow in $\vec{\mathcal{K}}_A$, we have the following data:

- $T=X\oplus \overline{T}$ with $X\in \mathrm{ind}(A)$,
- $T'=Y\oplus \overline{T}$ with $Y\in \mathrm{ind}(A)$,
- $\varepsilon: 0\to X\to M\to Y\to 0$ a non-split exact sequence in $\mathrm{mod}(A)$ with $M\in \mathrm{add}(\overline{T})$.

Thus, we only need to prove that $Y\in \mathrm{mod}_1(A)$ in order to get $T'\in \mathrm{mod}_1(A)$. For this purpose, we need the following lemma.

Lemma 3.2. *Let $\varepsilon: 0\to X\xrightarrow{u} M\to Y\to 0$ be an exact sequence in* $\mathrm{mod}(A)$ *verifying the following hypotheses*:

- $X, Y\in \mathrm{ind}(A)$ *and* $X=F_\lambda\widehat{X}$ *(with* $\widehat{X}\in \mathrm{ind}(\mathcal{C})$*),*
- $M=M_1\oplus\cdots\oplus M_t$ *where* $M_i=F_\lambda\widehat{M}_i\in \mathrm{ind}(A)$ *(with* $\widehat{M}_i\in \mathrm{ind}(\mathcal{C})$*), for every i,*
- $\mathrm{Ext}^1_A(Y,M)=0$.

Then (ε) is isomorphic to an exact sequence in $\mathrm{mod}(A)$:

$$0\to X\xrightarrow{\begin{bmatrix}F_\lambda u'_1\\ \vdots\\ F_\lambda u'_t\end{bmatrix}} M_1\oplus\cdots\oplus M_t\to Y\to 0,$$

where $u'_i\in \mathrm{Hom}_{\mathcal{C}}(\widehat{X}, {}^{g_i}\widehat{M}_i)$ for some $g_i\in G$, for every i.

Proof of Lemma 3.2. For short, we shall say that $u\in \mathrm{Hom}_A(X,M_i)$ is homogeneous of degree $g\in G$ if and only if $u=F_\lambda u'$ with $u'\in \mathrm{Hom}_{\mathcal{C}}(\widehat{X}, {}^{g}\widehat{M}_i)$. Recall from Section 1 that any

$u \in \mathrm{Hom}_A(X, M_i)$ is (uniquely) the sum of d homogeneous morphisms of pairwise different degrees (with $d \geqslant 0$). Let us write

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_t \end{bmatrix}$$

with $u_i : X \to M_i$ for each i. We may assume that $u_1 : X \to M_1$ is not homogeneous. Thus:

$$u_1 = h_1 + \cdots + h_d,$$

where $d \geqslant 2$ and $h_1, \ldots, h_d : X \to M_1$ are non-zero homogeneous morphisms of pairwise different degree. In order to prove the lemma, it suffices to prove the following property which we denote by $(\mathcal{P})$:

"(ε) *is isomorphic to an exact sequence of the form*:

$$0 \to X \xrightarrow{\begin{bmatrix} u_1' \\ u_2 \\ \vdots \\ u_t \end{bmatrix}} M_1 \oplus \cdots \oplus M_t \to Y \to 0, \tag{ε'}$$

where u_1' is the sum of at most $d-1$ non-zero homogeneous morphisms of pairwise different degree."

For simplicity we adopt the following notations:

- $\overline{M} = M_2 \oplus \cdots \oplus M_t$ (so $M = M_1 \oplus \overline{M}$),
- $\bar{u} = \begin{bmatrix} u_2 \\ \vdots \\ u_t \end{bmatrix} : X \to \overline{M} \left(\text{so } u = \begin{bmatrix} u_1 \\ \bar{u} \end{bmatrix} : X \to M_1 \oplus \overline{M} \right)$,
- $\bar{h} = h_2 + \cdots + h_d : X \to M_1$ (so $u_1 = h_1 + \bar{h}$).

Apply the functor $\mathrm{Hom}_A(-, M_1)$ to ε. Then, we get the exact sequence:

$$\mathrm{Hom}_A(M_1 \oplus \overline{M}, M_1) \xrightarrow{\mathrm{Hom}(u, M_1)} \mathrm{Hom}_A(X, M_1) \to \mathrm{Ext}^1_A(Y, M_1) = 0.$$

So there exists $[\lambda, \mu] : M_1 \oplus \overline{M} \to M_1$ such that $h_1 = [\lambda, \mu]u$. Hence:

$$h_1 = \lambda u_1 + \mu \bar{u} = \lambda h_1 + \lambda \bar{h} + \mu \bar{u}. \tag{i}$$

Let us distinguish two cases according to whether $\lambda \in \mathrm{End}_A(M_1)$ is invertible or nilpotent (recall that $M_1 \in \mathrm{ind}(A)$):

- If λ is invertible then:

$$\theta := \begin{bmatrix} \lambda & \mu \\ 0 & \mathrm{Id}_{\overline{M}} \end{bmatrix} : M_1 \oplus \overline{M} \to M_1 \oplus \overline{M}$$

is invertible. Using (i) we deduce an isomorphism of exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "{\left[\begin{smallmatrix} u_1 \\ \bar{u} \end{smallmatrix}\right]}"] \arrow[d, equal] & M_1 \oplus \overline{M} \arrow[r] \arrow[d, "\theta"] & Y \arrow[r] \arrow[d, "\sim"] & 0 & (\varepsilon) \\ {[-3pt]0} \arrow[r] & X \arrow[r, "{\left[\begin{smallmatrix} h_1 \\ \bar{u} \end{smallmatrix}\right]}"] & M_1 \oplus \overline{M} \arrow[r] & Y \arrow[r] & 0 & (\varepsilon'). \end{tikzcd}$$

Since $h_1 : X \to M_1$ is homogeneous, (ε') fits property $(\mathcal{P})$. So $(\mathcal{P})$ is satisfied in this case.

- If $\lambda \in \mathrm{End}_A(M_1)$ is nilpotent, let $p \geqslant 0$ be such that $\lambda^p = 0$. Using (i) we get the following equalities:

$$\begin{aligned} h_1 &= \lambda^2 h_1 + \left(\lambda^2 + \lambda\right)\bar{h} + (\lambda + \mathrm{Id}_{M_1})\mu\bar{u} \\ \vdots\ &\ \vdots\ \vdots \\ h_1 &= \lambda^t h_1 + \left(\lambda^t + \lambda^{t-1} + \cdots + \lambda\right)\bar{h} + \left(\lambda^{t-1} + \cdots + \lambda + \mathrm{Id}_{M_1}\right)\mu\bar{u} \\ \vdots\ &\ \vdots\ \vdots \\ h_1 &= \lambda^p h_1 + \left(\lambda^p + \lambda^{p-1} + \cdots + \lambda\right)\bar{h} + \left(\lambda^{p-1} + \cdots + \lambda + \mathrm{Id}_{M_1}\right)\mu\bar{u}. \end{aligned}$$

Since $\lambda^p = 0$ and $u_1 = h_1 + \bar{h}$ we infer that:

$$u_1 = \lambda'\bar{h} + \lambda'\mu\bar{u},$$

where $\lambda' := \mathrm{Id}_{M_1} + \lambda + \cdots + \lambda^{p-1} \in \mathrm{End}_A(M_1)$ is invertible. So we have an isomorphism:

$$\theta := \begin{bmatrix} \lambda' & \lambda'\mu \\ 0 & \mathrm{Id}_{\overline{M}} \end{bmatrix} : M_1 \oplus \overline{M} \to M_1 \oplus \overline{M}.$$

Consequently we have an isomorphism of exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "{\left[\begin{smallmatrix} \bar{h} \\ \bar{u} \end{smallmatrix}\right]}"] \arrow[d, equal] & M_1 \oplus \overline{M} \arrow[r] \arrow[d, "\theta"] & Y \arrow[r] \arrow[d, "\sim"] & 0 & (\varepsilon') \\ {[-3pt]0} \arrow[r] & X \arrow[r, "{\left[\begin{smallmatrix} u_1 \\ \bar{u} \end{smallmatrix}\right]}"] & M_1 \oplus \overline{M} \arrow[r] & Y \arrow[r] & 0 & (\varepsilon), \end{tikzcd}$$

where $\bar{h} = h_2 + \cdots + h_p$ is the sum of $p - 1$ non-zero homogeneous morphisms of pairwise different degrees. So $(\mathcal{P})$ is satisfied in this case. This finishes the proof of the lemma. □

Now we can prove that $Y \in \mathrm{mod}_1(A)$. Thanks to the preceding lemma, and with the same notations, we know that (ε) is isomorphic to an exact sequence in $\mathrm{mod}(A)$:

$$0 \to X \xrightarrow{\begin{bmatrix} F_\lambda u'_1 \\ \vdots \\ F_\lambda u'_t \end{bmatrix}} M_1 \oplus \cdots \oplus M_t \to Y \to 0, \tag{ε'}$$

where $u'_i \in \mathrm{Hom}_{\mathcal{C}}(\widehat{X}, {}^{g_i}\widehat{M}_i)$ for some $g_i \in G$, for every i. Therefore (recall that F_λ is exact):

$$Y \simeq \mathit{Coker} \begin{bmatrix} F_\lambda(u'_1) \\ \vdots \\ F_\lambda(u'_t) \end{bmatrix} \simeq F_\lambda \left(\mathit{Coker} \begin{bmatrix} u'_1 \\ \vdots \\ u'_t \end{bmatrix} \right).$$

This proves that $Y \in \mathrm{mod}_1(A)$. Therefore $T' = Y \oplus \overline{T} \in \mathrm{mod}_1(A)$.

To prove the opposite implication, that is $T \in \mathrm{mod}_1(A)$ if $T' \in \mathrm{mod}_1(A)$, we proceed similarly, except that instead of using Lemma 3.2 we use a dual version:

Lemma 3.3. *Let* $\varepsilon : 0 \to X \to M \xrightarrow{v} Y \to 0$ *be an exact sequence in* $\mathrm{mod}(A)$ *verifying the following hypotheses*:

- $X, Y \in \mathrm{ind}(A)$ *and* $Y = F_\lambda \widehat{Y}$ (*with* $\widehat{Y} \in \mathrm{ind}(\mathcal{C})$),
- $M = M_1 \oplus \cdots \oplus M_t$ *where* $M_i = F_\lambda \widehat{M}_i \in \mathrm{ind}(A)$ (*with* $\widehat{M}_i \in \mathrm{ind}(\mathcal{C})$), *for every* i,
- $\mathrm{Ext}^1_A(M, X) = 0$.

Then (ε) *is isomorphic to an exact sequence in* $\mathrm{mod}(A)$:

$$0 \to X \to M_1 \oplus \cdots \oplus M_T \xrightarrow{\begin{bmatrix} F_\lambda v'_1 \\ \vdots \\ F_\lambda v'_t \end{bmatrix}} Y \to 0,$$

where $v'_i \in \mathrm{Hom}_{\mathcal{C}}({}^{g_i}\widehat{M}_i, \widehat{Y})$ *for some* $g_i \in G$, *for every* i.

This finishes the proof of Proposition 3.1. □

Remark 3.4. Proposition 3.1 is similar to part of [13, Theorem 3.6] where P. Gabriel proves the following: if $F : \mathcal{C} \to A$ is a Galois covering with group G, with $\mathcal{C}$ locally bounded and such that G acts freely on $\mathrm{ind}(\mathcal{C})$, then for any connected component C of the Auslander–Reiten quiver of A, all indecomposable modules of C lie in $\mathrm{ind}_1(A)$ as soon as any one of them does.

Remark 3.5. The proof of Proposition 3.1 shows that for an arrow $T \to T'$ in $\vec{\mathcal{K}}_A$ such that $T, T' \in \mathrm{mod}_1(A)$ there exists an exact sequence in $\mathrm{mod}(\mathcal{C})$:

$$0 \to X \xrightarrow{\iota} M \xrightarrow{\pi} Y \to 0,$$

with the following properties:

- $T = F_\lambda X \oplus \overline{T}$ and $F_\lambda X \in \operatorname{ind}(A)$,
- $T' = F_\lambda Y \oplus \overline{T}$ and $F_\lambda Y \in \operatorname{ind}(A)$,
- $F_\lambda M \in \operatorname{add}(\overline{T})$.

Recall that $\vec{\mathcal{K}}_A$ has a Brauer–Thrall type property (see [18, Corollary 2.2]): $\vec{\mathcal{K}}_A$ is finite and connected if it has a finite connected component. In particular, $\vec{\mathcal{K}}_A$ is finite and connected if A is of finite representation type. Using Proposition 3.1, we get the following corollary.

Corollary 3.6. *If $\vec{\mathcal{K}}_A$ is finite (for example, A is of finite representation type), then any $T \in \vec{\mathcal{K}}_A$ is of the first kind with respect to F.*

Now we turn to the second goal of this section: for $T \in \operatorname{mod}_1(A)$ a basic tilting A-module, give sufficient conditions for $F.T$ to be a basic $\mathcal{C}$-module.

Proposition 3.7. *Let $T, T' \in \vec{\mathcal{K}}_A \cap \operatorname{mod}_1(A)$ lie in a same connected component of $\vec{\mathcal{K}}_A$, then:*

$$F.T \text{ is a basic } \mathcal{C}\text{-module if and only if } F.T' \text{ is a basic } \mathcal{C}\text{-module}.$$

In particular, if $T' = A$ or $T' = DA$, then $T \in \operatorname{mod}_1(A)$ and $F.T$ is a basic $\mathcal{C}$-module.

Proof. The k-category $\mathcal{C}$ is locally bounded so $F.A \simeq \mathcal{C}$ and $F.(DA) \simeq D\mathcal{C}$ are basic $\mathcal{C}$-modules. Therefore, we only need to prove the equivalence of the proposition. Without loss of generality, we may assume that there is an arrow $T \to T'$ in $\vec{\mathcal{K}}_A$. Let us assume that $F.T$ is basic and let us prove that so is $F.T'$. We use Remark 3.5 from which the adopt the notations, in particular, the exact sequence $0 \to X \xrightarrow{\iota} M \xrightarrow{\pi} Y \to 0$ in $\operatorname{mod}(\mathcal{C})$ is denoted by (ε). Because $F.T$ is basic and because of the properties verified by (ε), we only need to prove that $G_Y = 1$. Let $\varphi : Y \to {}^gY$ be an isomorphism in $\operatorname{mod}(\mathcal{C})$ (with $g \in G$), and let us prove that $g = 1$. To do this we construct an isomorphism $\theta : X \to {}^gX$. Notice that the following holds for every $h \in G$:

$$\begin{cases} {}^hX,\ {}^hM \in \operatorname{add}(F.T), \\ {}^hY,\ {}^hM \in \operatorname{add}(F.T'). \end{cases}$$

Moreover, thanks to $T \in \vec{\mathcal{K}}_A$ and to $F_\lambda F.T = \bigoplus_{h\in G} T$ (which is true because T is of the first kind with respect to F, see Section 1), we have, for every $i \geqslant 1$:

$$\operatorname{Ext}^i_{\mathcal{C}}(F.T, F.T) \simeq \operatorname{Ext}^i_A\big(F_\lambda F.(T), T\big) \simeq \prod_{h\in G} \operatorname{Ext}^i_A(T, T) = 0.$$

In particular:

$$\operatorname{Ext}^1_{\mathcal{C}}\big({}^gM, X\big) = \operatorname{Ext}^1_{\mathcal{C}}\big(M, {}^gX\big) = 0. \tag{i}$$

Therefore, applying the functor $\operatorname{Hom}_{\mathcal{C}}(M, -)$ to the exact sequence $({}^g\varepsilon)$ gives the exact sequence:

$$\operatorname{Hom}_{\mathcal{C}}\big(M, {}^gM\big) \xrightarrow{({}^g\pi)_*} \operatorname{Hom}_{\mathcal{C}}\big(M, {}^gY\big) \to \operatorname{Ext}^1_{\mathcal{C}}\big(M, {}^gX\big) = 0.$$

From this exact sequence, we deduce the existence of $\psi \in \mathrm{Hom}_{\mathcal{C}}(M, {}^gM)$ such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\pi"] \arrow[d, "\psi"'] & Y \arrow[d, "\varphi"] \\ {}^gM \arrow[r, "{}^g\pi"] & {}^gY. \end{tikzcd}$$

This implies the existence of $\theta \in \mathrm{Hom}_{\mathcal{C}}(X, {}^gX)$ making commutative the following diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "\iota"] \arrow[d, "\theta"'] & M \arrow[r, "\pi"] \arrow[d, "\psi"'] & Y \arrow[r] \arrow[d, "\varphi"'] & 0 \\ 0 \arrow[r] & {}^gX \arrow[r, "{}^g\iota"] & {}^gM \arrow[r, "{}^g\pi"] & {}^gY \arrow[r] & 0. \end{tikzcd} \tag{ii}$$

We claim that $\theta : X \to {}^gX$ is an isomorphism. The arguments that have been used to get (ii) may be adapted (just apply the functor $\mathrm{Hom}_{\mathcal{C}}({}^gM, -)$ to the exact sequence (ε) instead of applying the functor $\mathrm{Hom}_{\mathcal{C}}(M, -)$ to the exact sequence $({}^g\varepsilon)$, and use $\varphi^{-1} : {}^gY \to Y$ instead of using $\varphi : Y \to {}^gY$) to get the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & {}^gX \arrow[r, "{}^g\iota"] \arrow[d, "\theta'"'] & {}^gM \arrow[r, "{}^g\pi"] \arrow[d, "\psi'"'] & {}^gY \arrow[r] \arrow[d, "\varphi^{-1}"'] & 0 \\ 0 \arrow[r] & X \arrow[r, "\iota"] & M \arrow[r, "\pi"] & Y \arrow[r] & 0. \end{tikzcd} \tag{iii}$$

In order to show that $\theta : X \to {}^gX$ is an isomorphism, let us show that $\theta'\theta \in \mathrm{End}_{\mathcal{C}}(X)$ is an isomorphism. Notice that (ii) and (iii) give the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "\iota"] \arrow[d, "\theta'\theta-\mathrm{id}_X"'] & M \arrow[r, "\pi"] \arrow[d, "\psi'\psi-\mathrm{Id}_M"'] & Y \arrow[r] \arrow[d, "0"'] & 0 \\ 0 \arrow[r] & X \arrow[r, "\iota"] & M \arrow[r, "\pi"] & Y \arrow[r] & 0. \end{tikzcd} \tag{iv}$$

In particular we have $\pi(\psi'\psi - \mathrm{Id}_M) = 0$, so there exists $\lambda \in \mathrm{Hom}_{\mathcal{C}}(M, X)$ such that:

$$\psi'\psi - \mathrm{Id}_M = \iota\lambda.$$

Therefore:

$$\iota(\theta'\theta - \mathrm{Id}_X) = \iota\lambda\iota.$$

Since ι is one-to-one, we get $\theta'\theta - \mathrm{id}_X = \lambda\iota$, that is:

$$\theta'\theta = \mathrm{Id}_X + \lambda\iota.$$

If $\lambda\iota \in \mathrm{End}_{\mathcal{C}}(X)$ was an isomorphism, then $\iota : X \to M$ would be a section. This would imply that $F_\lambda X$ is a direct summand of $F_\lambda M$. This last property is impossible because: $T = F_\lambda X \oplus \overline{T}$, $F_\lambda M \in \mathrm{add}(\overline{T})$ and T is basic. This contradiction proves that $\lambda\iota \in \mathrm{End}_{\mathcal{C}}(X)$ is nilpotent. Therefore $\theta'\theta = \mathrm{Id}_X + \lambda\iota \in \mathrm{End}_{\mathcal{C}}(X)$ is invertible. As a consequence, $\theta : X \to {}^gX$ is a section. Since X, ${}^gX \in \mathrm{ind}(\mathcal{C})$, we deduce that $\theta : X \to {}^gX$ is an isomorphism. But we assumed that $F.T$ is basic, so $g = 1$. This finishes the proof of the implication:

$$\text{If } F.T \text{ is basic, then } F.T' \text{ is basic.}$$

After exchanging the roles of T and T' in the above arguments, we also prove that:

$$F.T \text{ is basic if } F.T' \text{ is basic}$$

under the assumption that $T \to T'$ is an arrow in $\vec{\mathcal{K}}_A$. This completes the proof of the proposition. $\square$

Proposition 3.7 has the following corollary which will be useful to prove Theorem 1.

Corollary 3.8. *Let $F : \mathcal{C} \to A$ be a connected Galois covering with group G. Let $T = T_1 \oplus \cdots \oplus T_n$ ($T_i \in \mathrm{ind}(A)$) and $T' = T'_1 \oplus \cdots \oplus T'_n$ ($T'_i \in \mathrm{ind}(A)$) be basic tilting A-modules lying in a same connected component of $\vec{\mathcal{K}}_A$. Assume that $T, T' \in \mathrm{mod}_1(A)$ and fix isomorphisms $\lambda_i : F_\lambda \widehat{T_i} \xrightarrow{\sim} T_i$ and $\lambda'_i : F_\lambda \widehat{T'}_i \xrightarrow{\sim} T'_i$ with $\widehat{T_i}, \widehat{T'}_i \in \mathrm{ind}(\mathcal{C})$ for every i. Then*:

$$F_{\widehat{T_i},\lambda_i} \textit{ is connected if and only if } F_{\widehat{T'}_i,\lambda'_i} \textit{ is connected.}$$

In particular, if $T' = A$ or $T' = DA$, then $F_{\widehat{T_i},\lambda_i}$ is connected.

Proof. Recall that "the Galois covering $\mathcal{E} \to \mathcal{B}$ is connected" means $\mathcal{E}$ is connected and locally bounded. Thanks to Remark 2.1 and to Proposition 2.10, we know that $F_{\widehat{T_i},\lambda_i}$ (or $F_{\widehat{T'}_i,\lambda'_i}$) is connected if and only if $F.T$ (or $F.T'$, respectively) is a basic $\mathcal{C}$-module. The corollary is therefore a consequence of Proposition 3.7. $\square$

We finish with the last objective of the section: give "simple" conditions on $T \in \vec{\mathcal{K}}_A$ under which condition $(H_{A,T})$ is satisfied (see Lemma 2.5).

Proposition 3.9. *Let T, T' lie in a same connected component of $\vec{\mathcal{K}}_A$. Then*:

$$(H_{A,T}) \textit{ is satisfied if and only if } (H_{A,T'}) \textit{ is satisfied.}$$

In particular, if $T' = A$ or $T' = DA$, then $(H_{A,T})$ is satisfied.

Proof. Let $\psi : A \xrightarrow{\sim} A$ be an automorphism restricting to the identity map on objects. Let $x \in \mathrm{Ob}(A)$ and let ${}_?A_x : y \in \mathrm{Ob}(A) \mapsto {}_yA_x$ be the indecomposable projective A-module associated to x. Then, we have an isomorphism of A-modules:

$$
\begin{aligned}
{}_{?}A_x &\to \psi.{}_{?}A_x,\\
u \in {}_yA_x &\mapsto F(u) \in (\psi.{}_{?}A_x)(y) = {}_yA_x.
\end{aligned}
$$

So $(H_{A,A})$ is satisfied, and similarly $(H_{A,DA})$ is satisfied. Therefore, in order to prove the proposition, it suffices to prove that $(H_{A,T})$ is satisfied if and only if $(H_{A,T'})$ is satisfied, for any arrow $T \to T'$ in $\vec{\mathcal{K}}_A$. Assume that $T \to T'$ is an arrow in $\vec{\mathcal{K}}_A$ and that $(H_{A,T})$ is satisfied. So we have the data:

- $T = X \oplus \overline{T}$ with $X \in \mathrm{ind}(A)$,
- $T' = Y \oplus \overline{T}$ with $Y \in \mathrm{ind}(A)$,
- a non-split exact sequence $0 \to X \xrightarrow{u} M \to Y \to 0$ where $M \in \mathrm{add}(\overline{T})$ and where $u : X \to M$ is the minimal left $\mathrm{add}(\overline{T})$-approximation of X.

Notice that in order to prove that $(H_{A,T'})$ is satisfied, we only need to prove that $\psi.Y \simeq Y$. Since ψ is an automorphism, $\psi. : \mathrm{mod}(A) \to \mathrm{mod}(A)$ is an equivalence of abelian categories. Therefore, the sequence $0 \to \psi.X \xrightarrow{\psi.u} \psi.M \to \psi.Y \to 0$ is non-split exact and verifies: $\psi.M \in \mathrm{add}(\psi.\overline{T})$ and $\psi.u : \psi.X \to \psi.M$ is the minimal left $\mathrm{add}(\psi.\overline{T})$-approximation of $\psi.Y$. Moreover $\psi.X \simeq X$, $\psi.M \simeq M$ and $\psi.\overline{T} \simeq \overline{T}$ because $(H_{A,T})$ is satisfied. So, $\psi.Y$ is isomorphic to the cokernel of the minimal left $\mathrm{add}(\overline{T})$-approximation of X. This implies that $Y \simeq \psi.Y$. So $(H_{A,T'})$ is satisfied. The converse is dealt with using dual arguments. $\square$

4. Comparison of $\vec{\mathcal{K}}_A$ and $\vec{\mathcal{K}}_{\mathrm{End}_A(T)}$ for a tilting A-module T

Let T be a basic tilting A-module. Let $B = \mathrm{End}_A(T)$. In the preceding section, we have pointed out conditions of the form: "the A-module T lies in the connected component of $\vec{\mathcal{K}}_A$ containing A." Since our final objective (that is, to compare the Galois coverings of A and of B) is symmetrical between A and B, we ought to find sufficient conditions under which the two following properties are verified:

- ${}_AT$ lies in the connected component of $\vec{\mathcal{K}}_A$ containing A,
- ${}_BT$ lies in the connected component of $\vec{\mathcal{K}}_B$ containing B.

Thus, this section is devoted to compare $\vec{\mathcal{K}}_A$ and $\vec{\mathcal{K}}_B$. For simplicity, if $X \in \mathrm{mod}(A)$ (or $u \in \mathrm{Hom}_A(X, Y)$) we shall write X_T (or u_T) for the B-module $\mathrm{Hom}_A(X, T)$ (or for the morphism of B-modules $\mathrm{Hom}_A(u, T) : \mathrm{Hom}_A(Y, T) \to \mathrm{Hom}_A(X, T)$, respectively). Also, whenever f is a morphism of modules, we shall write f_* (or f^*) for the mapping $g \mapsto fg$ (or for the mapping $g \mapsto gf$, respectively). We begin with a useful lemma.

Lemma 4.1. *Let $X \in \mathrm{mod}(A)$ and let $T' \in \vec{\mathcal{K}}_A$ be a predecessor of T (that is, there is an oriented path in $\vec{\mathcal{K}}_A$ starting at T' and ending at T). Then, there is an isomorphism, for any $Y \in \mathrm{add}(T')$:*

$$
\begin{aligned}
\theta_{X,Y} : \mathrm{Hom}_A(X, Y) &\to \mathrm{Hom}_B(Y_T, X_T),\\
u &\mapsto u_T.
\end{aligned}
$$

In particular: Y lies in $\mathrm{ind}(A)$ if and only if Y_T lies in $\mathrm{ind}(B)$, for any $Y \in \mathrm{add}(T')$.

Proof. Remark that $\theta_{X,T'}$ is an isomorphism if and only if $\theta_{X,Y}$ is an isomorphism for any $Y \in \mathrm{add}(T')$. By assumption on T', there exists a path in $\vec{\mathcal{K}}_A$ starting at T' and ending at T. Let us prove by induction on the length l of this path that $\theta_{X,T'}$ is an isomorphism.

If $l = 0$ then $T = T'$. So $\theta_{X,T'} = \theta_{X,T}$ is equal to:

$$\mathrm{Hom}_A(X, T) = X_T \to \mathrm{Hom}_B(T_T, X_T) = \mathrm{Hom}_B(B, X_T),$$
$$u \mapsto (f \mapsto fu).$$

So $\theta_{X,T'}$ is an isomorphism (with inverse $\varphi \mapsto \varphi(1_B)$). This proves the lemma when $l = 0$.

Now, assume that $l > 0$ and assume that $\theta_{X,T''}$ is an isomorphism whenever T'' is the source of a path in $\vec{\mathcal{K}}_A$ ending at T and with length equal to $l - 1$. We have a path $T' \to T'' \to \cdots \to T$ of length l in $\vec{\mathcal{K}}_A$. Therefore:

$$\theta_{X,Y} \text{ is an isomorphism for any } Y \in \mathrm{add}(T''). \tag{i}$$

Moreover, thanks to the arrow $T' \to T''$ in $\vec{\mathcal{K}}_A$, we have:

(ii) $T' = \overline{T} \oplus Y'$ with $Y' \in \mathrm{ind}(A)$,
(iii) $T'' = \overline{T} \oplus Y''$ with $Y'' \in \mathrm{ind}(A)$,
(iv) a non-split exact sequence $0 \to Y' \to M \to Y'' \to 0$ with $M \in \mathrm{add}(\overline{T})$.

Thanks to (i), (ii) and (iii) we only need to prove that $\theta_{X,Y'}$ is an isomorphism. Remark that by assumption on T' and T'' we have $T \in T^{\perp} \subseteq T''^{\perp}$. This implies in particular that $\mathrm{Ext}^1_A(Y'', T) = 0$. Therefore, (iv) yields an exact sequence in $\mathrm{mod}(A)$:

$$0 \to Y''_T \to M_T \to Y'_T \to 0.$$

This gives rise to the exact sequence:

$$0 \to \mathrm{Hom}_B\big(Y'_T, X_T\big) \to \mathrm{Hom}_B(M_T, X_T) \to \mathrm{Hom}_B\big(Y''_T, X_T\big).$$

On the other hand, (iv) yields the following exact sequence:

$$0 \to \mathrm{Hom}_A(X, Y') \to \mathrm{Hom}_A(X, M) \to \mathrm{Hom}_A(X, Y'').$$

Therefore, we have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(X, Y') \arrow[r] \arrow[d, "\theta_{X,Y'}"'] & \mathrm{Hom}_A(X, M) \arrow[r] \arrow[d, "\theta_{X,M}"'] & \mathrm{Hom}_A(X, Y'') \arrow[d, "\theta_{X,Y''}"'] \\
0 \arrow[r] & \mathrm{Hom}_A(Y'_T, X_T) \arrow[r] & \mathrm{Hom}_A(M_T, X_T) \arrow[r] & \mathrm{Hom}_A(Y''_T, X_T),
\end{tikzcd}$$

where the rows are exact and where $\theta_{X,M}$ and $\theta_{X,Y''}$ are isomorphisms. This shows that $\theta_{X,Y'}$ is an isomorphism. So $\theta_{X,T'}$ is an isomorphism and the induction is finished. This proves the first assertion of the lemma. The second assertion is due to the functoriality of $\theta_{X,Y}$. $\square$

Remark 4.2. Assume that A is hereditary. Then Lemma 4.1 still holds if one replaces the hypothesis "T' is a predecessor of T" by "$T' \geqslant T$" (that is, $T^{\perp} \subseteq T'^{\perp}$). The proof is then a classical application of left $\mathrm{add}(T)$-approximations.

The following proposition is the base of the link between $\vec{\mathcal{K}}_A$ and $\vec{\mathcal{K}}_B$: it explains how to associate suitable tilting B-modules with tilting A-modules.

Proposition 4.3. *Let $X \to Y$ be an arrow in $\vec{\mathcal{K}}_A$ where X and Y are predecessors of T. Then*:

$$X_T \in \vec{\mathcal{K}}_B \text{ if and only if } Y_T \in \vec{\mathcal{K}}_B.$$

If the two conditions of the above equivalence are satisfied, then there is an arrow $Y_T \to X_T$ in $\vec{\mathcal{K}}_B$.

Proof. Let us assume that $Y_T \in \vec{\mathcal{K}}_B$ and let us show that $X_T \in \vec{\mathcal{K}}_B$ and that there is an arrow $Y_T \to X_T$ in $\vec{\mathcal{K}}_B$ (the proof of the remaining implication is then obtained by exchanging the roles of X and Y). The arrow $X \to Y$ in $\vec{\mathcal{K}}_A$ gives the following data:

- $X = M \oplus \overline{X}$ with $M \in \mathrm{ind}(A)$,
- $Y = N \oplus \overline{X}$ with $N \in \mathrm{ind}(A)$,
- $\varepsilon : 0 \to M \xrightarrow{i} X' \xrightarrow{p} N \to 0$ is a non-split exact sequence in $\mathrm{mod}(A)$ with $X' \in \mathrm{add}(\overline{X})$.

The tilting A-module Y is a predecessor of T. Hence $T \in T^{\perp} \subseteq Y^{\perp}$ and therefore $\mathrm{Ext}^1_A(N, T) = 0$. Applying the functor $\mathrm{Hom}_A(-, T)$ to the exact sequence (ε), we get an exact sequence in $\mathrm{mod}(B)$:

$$0 \to N_T \xrightarrow{p_T} X'_T \xrightarrow{i_T} M_T \to 0. \qquad (\varepsilon_T)$$

Notice that we also have:

- $X_T = M_T \oplus \overline{X}_T$,
- $Y_T = N_T \oplus \overline{X}_T$,
- $X'_T \in \mathrm{add}(\overline{X}_T)$.

Therefore, in order to prove that $X_T \in \vec{\mathcal{K}}_B$ and that there is an arrow $Y_T \to X_T$ in $\vec{\mathcal{K}}_B$, we only need to prove the following facts:

(1) ε_T does not split,
(2) $M_T \in \mathrm{ind}(B)$ and $N_T \in \mathrm{ind}(B)$,
(3) $\mathrm{pd}_B(X_T) < \infty$,
(4) X_T is selforthogonal,
(5) X_T is the direct sum of n indecomposable A-modules and X_T is basic.

(1) Let us prove that ε_T does not split. If ε_T splits, then i_T is a retraction, so that there exists $\lambda \in \mathrm{Hom}_B(M_T, X'_T)$, verifying $\mathrm{Id}_{M_T} = i_T \circ \lambda$. Since M is a direct summand of $X \in \vec{\mathcal{K}}_A$ and since X is a predecessor of T, Lemma 4.1 implies that $\lambda = \pi_T$ with $\pi \in \mathrm{Hom}_A(X', M)$. Thus we

have $(\pi \circ i)_T = (\mathrm{Id}_M)_T$. Using again Lemma 4.1 we deduce that $\pi \circ i = \mathrm{Id}_M$ which is impossible because ε does not split. So ε_T does not split.

(2) Lemma 4.1 implies that $M_T, N_T \in \mathrm{ind}(B)$.

(3) Since we assumed that $Y_T \in \vec{\mathcal{K}}_B$, we have $\mathrm{pd}_B(\overline{X}_T) < \infty$, $\mathrm{pd}_B(X'_T) < \infty$ and $\mathrm{pd}_B(N_T) < \infty$. Hence ε_T gives $\mathrm{pd}_B(M_T) < \infty$. So $\mathrm{pd}_B(X_T) < \infty$.

(4) Let us prove that X_T is selforthogonal. For this purpose, we use the following lemma.

Lemma 4.4. *Let $L \in \mathrm{add}(X)$. Then, the following morphism induced by $p_T : N_T \to X'_T$ is surjective*:

$$\begin{aligned}(p_T)^* : \mathrm{Hom}_B\big(X'_T, L_T\big) &\to \mathrm{Hom}_B(N_T, L_T),\\ f &\mapsto f \circ p_T.\end{aligned}$$

Proof. Since $L \in \mathrm{add}(X)$ and since $X \in \vec{\mathcal{K}}_A$, we have $\mathrm{Ext}^1_A(L, M) = 0$. Hence, the exact sequence obtained by applying $\mathrm{Hom}_A(L, -)$ to ε gives rise to a surjective morphism induced by p:

$$\begin{aligned}p_* : \mathrm{Hom}_A(L, X') &\twoheadrightarrow \mathrm{Hom}_A(L, N),\\ f &\mapsto p \circ f.\end{aligned}$$

Let us apply Lemma 4.1 to $X' \in \mathrm{add}(Y)$ and to $N \in \mathrm{add}(Y)$. We get the following commutative diagram where vertical arrows are isomorphisms:

$$\begin{tikzcd}
\mathrm{Hom}_A(L, X') \arrow[r, "p_*"] \arrow[d, "\theta_{L,X'}"'] & \mathrm{Hom}_A(L, N) \arrow[d, "\theta_{L,N}"'] \\
\mathrm{Hom}_B(X'_T, L_T) \arrow[r, "(p_T)^*"] & \mathrm{Hom}_B(N_T, L_T).
\end{tikzcd}$$

Since p_* is surjective, we infer that so is $(p_T)^*$. □

Now we can prove that $X_T = \overline{X}_T \oplus M_T$ is selforthogonal. Since $\overline{X}_T \in \mathrm{add}(Y_T)$ and $Y_T \in \vec{\mathcal{K}}_B$, we get, for every $i \geqslant 1$:

$$\mathrm{Ext}^i_B(\overline{X}_T, \overline{X}_T) = 0. \tag{i}$$

For each $i \geqslant 1$, the functor $\mathrm{Hom}_B(\overline{X}_T, -)$ applied to ε_T gives rise to the following exact sequence:

$$\mathrm{Ext}^i_B\big(\overline{X}_T, X'_T\big) \to \mathrm{Ext}^i_B(\overline{X}_T, M_T) \to \mathrm{Ext}^{i+1}_B(\overline{X}_T, N_T).$$

Since $\overline{X}_T, X'_T, N_T \in \mathrm{add}(Y_T)$ and $Y_T \in \vec{\mathcal{K}}_B$, we get, for every $i \geqslant 1$:

$$\mathrm{Ext}^i_B(\overline{X}_T, M_T) = 0. \tag{ii}$$

On the other hand, the functor $\mathrm{Hom}_B(-, \overline{X}_T)$ applied to ε_T gives rise to the following exact sequences:

$$.\mathrm{Hom}_B\big(X'_T, \overline{X}_T\big) \xrightarrow{(p_T)^*} \mathrm{Hom}_B(N_T, \overline{X}_T) \to \mathrm{Ext}^1_B(M_T, \overline{X}_T) \to \mathrm{Ext}^1_B\big(X'_T, \overline{X}_T\big),$$
$$.\,\mathrm{Ext}^i_B(N_T, \overline{X}_T) \to \mathrm{Ext}^{i+1}_B(M_T, \overline{X}_T) \to \mathrm{Ext}^{i+1}_B\big(X'_T, \overline{X}_T\big) \quad \text{for } i \geqslant 1.$$

These exact sequences together with Lemma 4.4 and the selforthogonality of Y_T imply that for any $i \geqslant 1$:

$$\mathrm{Ext}^i_B(M_T, \overline{X}_T) = 0. \tag{iii}$$

In order to get the selforthogonality of $X_T = M_T \oplus \overline{X}_T$ it only remains to prove that M_T is selforthogonal (because of (i), (ii) and (iii)). Notice that the functor $\mathrm{Hom}_B(N_T, -)$ applied to ε_T gives rise to the following exact sequence for each $i \geqslant 1$:

$$\mathrm{Ext}^i_B\big(N_T, X'_T\big) \to \mathrm{Ext}^i_B(N_T, M_T) \to \mathrm{Ext}^{i+1}_B(N_T, N_T).$$

Using $Y_T \in \vec{\mathcal{K}}_B$ and $X'_T, N_T \in \mathrm{add}(Y_T)$ we deduce that, for every $i \geqslant 1$:

$$\mathrm{Ext}^i_B(N_T, M_T) = 0. \tag{iv}$$

Finally the functor $\mathrm{Hom}_B(-, M_T)$ applied to ε_T gives rise to the following exact sequences:

$$.\mathrm{Hom}_B\big(X'_T, M_T\big) \xrightarrow{(p_T)^*} \mathrm{Hom}_B(N_T, M_T) \to \mathrm{Ext}^1_B(M_T, M_T) \to \mathrm{Ext}^1_B\big(X'_T, M_T\big),$$
$$.\,\mathrm{Ext}^i_B(N_T, M_T) \to \mathrm{Ext}^{i+1}_B(M_T, M_T) \to \mathrm{Ext}^{i+1}_B\big(X'_T, M_T\big) \quad \text{for } i \geqslant 1.$$

These exact sequences together with Lemma 4.4, (ii) and (iv) imply that for every $i \geqslant 1$ (recall that $X'_T \in \mathrm{add}(\overline{X}_T)$):

$$\mathrm{Ext}^i_B(M_T, M_T) = 0. \tag{v}$$

From (i), (ii), (iii) and (v) we deduce that $X_T = M_T \oplus \overline{X}_T$ is selforthogonal.

(5) To finish, let us prove that X_T is basic and that X_T is the direct sum of n indecomposable modules. Notice that $\overline{X}_T$ is basic because it is a direct summand of the basic tilting B-module Y_T. On the other hand, ε_T does not split, so $\mathrm{Ext}^1_A(M_T, N_T) \neq 0$, hence $M_T \notin \mathrm{add}(Y_T)$ and therefore $M_T \notin \mathrm{add}(\overline{X}_T)$. Since $M_T \in \mathrm{ind}(B)$, we deduce that X_T is basic. Finally, Y_T is by assumption the direct sum of n indecomposable modules, and X_T and Y_T differ by one indecomposable direct summand so X_T is also the direct sum of n indecomposable modules. □

Remark 4.5. When A is hereditary, Proposition 4.3 has the following generalisation: *Let $X \in \vec{\mathcal{K}}_A$ be such that $X \geqslant T$, then $X_T \in \vec{\mathcal{K}}_B$.* The proof of this generalisation is obtained by replacing the use of the exact sequence ε by a coresolution of X in $\mathrm{add}(T)$.

Proposition 4.3 gives the following proposition which will be used in the comparison of the Galois coverings of A and B. We omit the proof which is immediate using Proposition 4.3.

Proposition 4.6. *Let $X \in \vec{\mathcal{K}}_A$ be such that there exists a path in $\vec{\mathcal{K}}_A$ starting at X and ending at T. Then $X_T \in \vec{\mathcal{K}}_B$ and there exists in $\vec{\mathcal{K}}_B$ a path starting at B and ending at X_T.*

Proposition 4.3 also allows us to prove the main result of this section. Recall that for a quiver Q, we write Q^{op} for the opposite quiver (obtained from Q by reversing the arrows).

Theorem 4.7. *Let $\vec{\mathcal{K}}_A(T)$ (or $\vec{\mathcal{K}}_B(T)$) be the convex hull of $\{A,T\}$ (or of $\{B,T\}$) in $\vec{\mathcal{K}}_A$ (or in $\vec{\mathcal{K}}_B$, respectively). Then we have an isomorphism of quivers:*

$$\begin{aligned} \alpha : \vec{\mathcal{K}}_A(T) &\mapsto \vec{\mathcal{K}}_B(T)^{op}, \\ X &\mapsto X_T = \mathrm{Hom}_A(X,T). \end{aligned}$$

Under this correspondence, $A \in \vec{\mathcal{K}}_A(T)$ (or the A-module $T \in \vec{\mathcal{K}}_A(T)$) is associated with the B-module $T \in \vec{\mathcal{K}}_B(T)$ (or with $B \in \vec{\mathcal{K}}_B(T)$, respectively).

Proof. Thanks to Proposition 4.3, the mapping α is a well-defined morphism of quivers. Thus, it only remains to exhibit an inverse morphism. Notice that Proposition 4.6 implies that $\vec{\mathcal{K}}_A(T) = \{A,T\}$ if and only if $\vec{\mathcal{K}}_B(T) = \{B,T\}$, if and only if there is no path in $\vec{\mathcal{K}}_A(T)$ starting at A and ending at T. Therefore, we may assume that there is a path starting at A and ending at T. This assumption implies that any $X \in \vec{\mathcal{K}}_A(T)$ is a predecessor of T. From [22, Theorem 1.5] we know that T is a basic tilting $\mathrm{End}_B(T)$-module and that we have an isomorphism of k-algebras:

$$\begin{aligned} A &\to \mathrm{End}_B(T), \\ a &\mapsto (t \mapsto at). \end{aligned}$$

Henceforth, we shall consider A-modules as $\mathrm{End}_B(T)$-modules and vice versa using the above isomorphism. In particular, we have an identification of quivers:

$$\begin{aligned} \vec{\mathcal{K}}_A(T) &\xrightarrow{\sim} \vec{\mathcal{K}}_{\mathrm{End}_B(T)}(T), \\ X &\mapsto X. \end{aligned}$$

Therefore, we also have a well-defined morphism of quivers:

$$\begin{aligned} \alpha' : \vec{\mathcal{K}}_B(T)^{op} &\to \vec{\mathcal{K}}_A(T), \\ X &\mapsto X_T = \mathrm{Hom}_B(X,T). \end{aligned}$$

Let us prove that $\alpha'\alpha$ is an isomorphism. Let $X \in \vec{\mathcal{K}}_A(T)$. Then X is a predecessor of T. Therefore, Lemma 4.1 implies that:

$$\mathrm{Hom}_B\big(\mathrm{Hom}_A(X,T),T\big) \simeq \mathrm{Hom}_B\big(\mathrm{Hom}_A(X,T),\mathrm{Hom}_A(A,T)\big) \simeq \mathrm{Hom}_A(A,X) \simeq X.$$

This proves that $\alpha'\alpha$ is an isomorphism of quivers. With the same arguments one also shows that $\alpha\alpha'$ is an isomorphism. So does $\alpha : \vec{\mathcal{K}}_A(T) \to \vec{\mathcal{K}}_B(T)^{op}$. $\square$

Notice that $\vec{\mathcal{K}}_A$ and $\vec{\mathcal{K}}_B^{op}$ are not isomorphic in general. Indeed these quivers may have different number of vertices as the following example shows.

Example 4.8. Let Q be the quiver:

$$\begin{tikzcd}
& 2 \arrow[dr] & \\
1 \arrow[ur] \arrow[rr] & & 3,
\end{tikzcd}$$

and let $A = kQ/I$ where I is the ideal generated by the oriented path of length 2 in Q. Notice that A is of finite representation type. Let $T = P_1 \oplus P_2 \oplus \tau_A^{-1} P_3$ be the APR-tilting A-module associated to the sink 3. Hence:

$$T = \begin{smallmatrix} & 1 & \\ 2 & & 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 1 & 2 \\ 2 & 3 \end{smallmatrix},$$

and the Hasse diagram $\vec{\mathcal{K}}_A$ of basic tilting A-modules is equal to:

$$\begin{tikzcd}
& & & \bullet \arrow[dr] & & \\
& & \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[r] & D(A). \\
A \arrow[r] & T \arrow[ur] \arrow[dr] & & \bullet \arrow[ur] & & \\
& & \bullet \arrow[ur] & & &
\end{tikzcd}$$

On the other hand, $B = \mathrm{End}_A(T)$ is isomorphic to kQ'/I' where Q' is equal to:

$$\begin{tikzcd}
a \arrow[r] & b \arrow[r, bend left] & c \arrow[l, bend left]
\end{tikzcd}$$

and I' is the ideal generated by the path $c \to b \to c$. As a B-module, T is equal to:

$$T = \begin{smallmatrix} a \\ b \\ c \\ b \end{smallmatrix} \oplus \begin{smallmatrix} c \\ b \end{smallmatrix} \oplus \begin{smallmatrix} a & & c \\ & b & \end{smallmatrix},$$

and $\vec{\mathcal{K}}_B$ is equal to:

$$
\begin{tikzcd}
 & & & \bullet \arrow[dr] & & \\
 & & \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[dr] & \\
 & T \arrow[ur] & & \bullet \arrow[ur] \arrow[dr] & & DB. \\
B \arrow[ur] \arrow[dr] & & \bullet \arrow[ur] & & \bullet \arrow[ur] & \\
 & \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[ur] & & \\
 & & \bullet \arrow[ur] & & &
\end{tikzcd}
$$

In particular, $\vec{\mathcal{K}}_A$ and $\vec{\mathcal{K}}_B$ do not have the same number of vertices. Notice that, in this example, the isomorphism of Theorem 4.7 is equal to:

$$
\begin{aligned}
\vec{\mathcal{K}}_A(T) = (A \to T) \to \vec{\mathcal{K}}_B(T)^{op} &= (T \to B), \\
A \mapsto T &= \mathrm{Hom}_A(A, T), \\
T \mapsto B &= \mathrm{Hom}_A(T, T).
\end{aligned}
$$

Remark 4.9. Assume that A is hereditary, then Theorem 4.7 has the following generalisation, thanks to Remark 4.5: *Let Q_A (or Q_B) be the full subquiver of $\vec{\mathcal{K}}_A$ (or of $\vec{\mathcal{K}}_B$, respectively) made of the tilting modules $X \geqslant T$. Then $X \mapsto X_T$ induces an isomorphism of quivers $Q_A \xrightarrow{\sim} Q_B^{op}$.*

5. Comparison of the Galois coverings of A and $\mathrm{End}_A(T)$ for T basic tilting A-module

This section is devoted to the proofs of Theorem 1, of Corollaries 1 and 2. Let $T \in \vec{\mathcal{K}}_A$ and let $B = \mathrm{End}_A(T)$. As in the introduction, we shall say that A and B have the same connected Galois coverings with group G if and only if there exists a bijection $\mathrm{Gal}_A(G) \xrightarrow{\sim} \mathrm{Gal}_B(G)$. Here $\mathrm{Gal}_A(G)$ denotes the set of equivalence classes of connected Galois coverings with group G of A. In order to compare the equivalence classes of connected Galois coverings of A and those of B, we introduce the following assertion which depends on A, on T and on a fixed group G:

$\mathcal{P}(A, T, G) =$ "$(H_{A,T})$ *is satisfied and for any connected Galois covering $F : \mathcal{C} \to A$ with group G, the A-module T is of the first kind with respect to F and $F.T$ is a basic $\mathcal{C}$-module.*"

Recall from Definition 2.7 that the condition $(H_{A,T})$ ensures the existence of an equivalence class $[F]_T$ of Galois coverings of B depending only on the equivalence class $[F]$ of F. Recall also from Remark 2.1 and from Proposition 2.10 that the condition "$F.T$ is a basic $\mathcal{C}$-module" implies that $[F]_T$ is an equivalence class of connected Galois coverings of B. Finally, recall that

$\mathcal{P}(A, A, G)$ and $\mathcal{P}(A, DA, G)$ are true for any G (see Propositions 3.1, 3.7 and 3.9). The above definition of $\mathcal{P}(A, T, G)$ is relevant because of the following proposition.

Proposition 5.1. *Let G be a group. Assume that $\mathcal{P}(A, T, G)$ and $\mathcal{P}(B, T, G)$ are true. Then A and B have the same connected Galois coverings with group G.*

Proof. Since $\mathcal{P}(A, T, G)$ is true, we have a well-defined mapping:

$$\begin{aligned} \varphi_A : \mathrm{Gal}_A(G) &\to \mathrm{Gal}_B(G), \\ [F] &\mapsto [F]_T. \end{aligned} \tag{i}$$

Similarly, $\mathcal{P}(B, T, G)$ is true so we have a well-defined mapping:

$$\begin{aligned} \varphi_B : \mathrm{Gal}_B(G) &\to \mathrm{Gal}_G\big(\mathrm{End}_B(T)\big), \\ [F] &\mapsto [F]_T. \end{aligned} \tag{ii}$$

Thanks to Proposition 2.11 we know that $\rho_A^{-1} \circ (\varphi_B \varphi_A([F])) = [F]$ for any $[F] \in \mathrm{Gal}_A(G)$. Therefore, φ_A is one-to-one and φ_B is onto. Notice that thanks to the isomorphism $\rho_A : A \xrightarrow{\sim} \mathrm{End}_B(T)$, the assertion $\mathcal{P}(\mathrm{End}_B(T), T, G)$ is true, so that the above arguments imply that φ_B is one-to-one and that $\varphi_{\mathrm{End}_B(T)}$ is onto. As a consequence, φ_B is bijective, so the mapping $[F] \mapsto \rho_A^{-1} \circ [F]_T$ induces a bijection $\mathrm{Gal}_B(G) \xrightarrow{\sim} \mathrm{Gal}_A(G)$. □

Thanks to Proposition 5.1 we are reduced to find sufficient conditions for $\mathcal{P}(A, T, G)$ and $\mathcal{P}(B, T, G)$ to be simultaneously true. The following proposition is a direct consequence of Proposition 3.1, of Corollary 3.8, of Proposition 3.9 and of the fact that $\mathcal{P}(A, A, G)$ and $\mathcal{P}(A, DA, G)$ are true.

Proposition 5.2. *Let G be a group. Let $T' \in \vec{\mathcal{K}}_A$ lying in the connected component of $\vec{\mathcal{K}}_A$ containing T. Then:*

$$\mathcal{P}(A, T, G) \textit{ is true if and only if } \mathcal{P}(A, T', G) \textit{ is true.}$$

In particular, if $T' = A$ or if $T' = DA$ then $\mathcal{P}(A, T, G)$ is true.

Thanks to Proposition 5.2, we just have to look for conditions for ${}_AT$ and ${}_BT$ to lie in the connected component of $\vec{\mathcal{K}}_A$ and $\vec{\mathcal{K}}_B$ containing A and B respectively. Such a condition is given by the following proposition.

Proposition 5.3. *Let G be a group and assume that there exists a path in $\vec{\mathcal{K}}_A$ starting at A and ending at T. Then ${}_AT$ and A lie in the same and connected component of $\vec{\mathcal{K}}_A$. Also, ${}_BT$ and B lie in the same connected component of $\vec{\mathcal{K}}_B$. Consequently, $\mathcal{P}(A, T, G)$ and $\mathcal{P}(B, T, G)$ are true.*

Proof. Theorem 4.7 implies that there exists a path in $\vec{\mathcal{K}}_B$ starting at $\mathrm{Hom}_A(T, T) = B$ and ending at $\mathrm{Hom}_A(A, T) = T$. Using Proposition 5.2 we get the desired conclusion. □

Now we can prove Theorem 1:

Proof of Theorem 1. (1) Since T and T' lie in a same connected component of $\vec{\mathcal{K}}_A$, there exists a sequence $T^{(1)} = T, T^{(2)}, \ldots, T^{(r)} = T'$ of basic tilting A-modules such that for any $i \in \{1, \ldots, r-1\}$, there exists a path in $\vec{\mathcal{K}}_A$ with $T^{(i)}$ and $T^{(i+1)}$ as end-points. For short, let us write B_i for $\mathrm{End}_A(T^{(i)})$. Let $i \in \{1, \ldots, r-1\}$ and let us assume, for example, that there exists a path in $\vec{\mathcal{K}}_A$ starting at $T^{(i)}$ and ending at $T^{(i+1)}$. Using Lemma 4.1 and Proposition 4.6 we infer that:

(i) $\mathrm{End}_A(T^{(i)})$ and $\mathrm{End}_{B_{i+1}}(\mathrm{Hom}_A(T^{(i)}, T^{(i+1)}))$ are isomorphic as k-algebras (and therefore as k-categories),
(ii) there exists a path in $\vec{\mathcal{K}}_{B_{i+1}}$ starting at B_{i+1} and ending at $\mathrm{Hom}_A(T^{(i)}, T^{(i+1)})$.

This implies (thanks to Propositions 5.3 and 5.1) that $\mathrm{End}_A(T^{(i)})$ and $\mathrm{End}_A(T^{(i+1)})$ have the same connected Galois coverings with group G. Since this fact is true for any i, we deduce that $\mathrm{End}_A(T)$ and $\mathrm{End}_A(T')$ have the same connected Galois coverings with group G.

(2) is a consequence of (1), of the fact that $\mathrm{End}_A(A) \simeq \mathrm{End}_A(DA) \simeq A^{op}$ and of the fact that A and A^{op} have the same Galois coverings ($F : \mathcal{C} \to A$ is a Galois covering if and only if $F^{op} : \mathcal{C}^{op} \to A$ is a Galois covering and $\mathcal{C}^{op}$ is connected and locally bounded if and only if $\mathcal{C}^{op}$ is). $\square$

Using Theorem 1 we can prove Corollaries 1 and 2.

Proof of Corollary 1. Since A is of finite representation type, Theorem 1 implies that A and B have the same connected Galois coverings. Since we are dealing with representation-finite standard algebras, A (or B) admits a connected Galois covering with group G if and only if G is a factor group of the fundamental group $\pi_1(A)$ (or $\pi_1(B)$) of the Auslander–Reiten quiver of A (or of B, respectively). Consequently, $\pi_1(A)$ and $\pi_1(B)$ are isomorphic. $\square$

Proof of Corollary 2. (1) and (2) are consequences of Theorem 1 and of the fact that A is simply connected if and only if it has no proper connected Galois covering (see [20, Corollary 4.5]).

(3) is a consequence of (2). $\square$

Corollary 1 naturally leads to the following question: let G be a group such that A and B have the same Galois coverings with group G, is it true that A admits an admissible presentation (that is, a presentation by quiver and admissible relations) with fundamental group isomorphic to G if and only if the same holds for B? The answer is no in general as the following example shows:

Example 5.4. Let Q be the following quiver:

$$\begin{tikzcd} & 2 \arrow[dr, "b"] & & \\ 1 \arrow[ur, "b"] \arrow[rr, "a"'] & & 3 \arrow[r, "d"'] & 4, \end{tikzcd}$$

and let $A = kQ/I$ where $I = \langle da \rangle$. Let $T = P_1 \oplus P_2 \oplus P_3 \oplus \tau_A^{-1}(P_4) = P_1 \oplus P_2 \oplus P_3 \oplus S_3$ be the APR-tilting module associated with the sink 3 (here S_i is the simple A-module associated to the vertex i and P_i is the indecomposable projective A-module with top S_i). Then $B = \mathrm{End}_A(T)$ is the path algebra of the following quiver:

$$\begin{tikzcd} & {} \arrow[dr] & \\ {} \arrow[ur] \arrow[dr] & & {} . \\ & {} \arrow[ur] & \end{tikzcd}$$

Since T is an APR-tilting A-module, there is an arrow $A \to T$ in $\vec{\mathcal{K}}_A$. Then, Theorem 1 implies that for any group G, the k-algebras A and B have the same connected Galois covering with group G. On the other hand, any admissible presentation of B has fundamental group isomorphic to $\mathbb{Z}$ (see [8, Theorem 3.5]) whereas A admits an admissible presentation with fundamental group 0 and another one with fundamental group isomorphic to $\mathbb{Z}$ (see [2, 1.4] for example).

In the preceding example, the reader may remark that the fundamental group of any admissible presentation of A is a factor group of $\mathbb{Z}$ and that the same holds for B. Let us say that A admits an optimum fundamental group (G) if and only if there exists an admissible presentation of A with fundamental group G and if the fundamental group of any other admissible presentation is a factor group of G. For example, A admits an optimum fundamental group in the following cases: A is of finite representation type and standard (see [13]), A is constricted (see [8, Theorem 3.5]), A is monomial, A is triangular and has no double bypass (see [20, Theorem 1]). Then we have the following corollary whose proof is a direct consequence of Theorem 1:

Corollary 5.5. *Assume that T lies in the connected component of $\vec{\mathcal{K}}_A$ containing A. Then A admits G as optimum fundamental group if and only if B admits G as optimum fundamental group.*

6. On the simple connectedness of a tilted algebra

The aim of this section is to prove Proposition 1. We shall use the construction of the preceding section. Recall from Remark 2.1, Lemma 2.2 and Proposition 2.10 that given a tilting A-module T and a connected Galois covering $F : \mathcal{C} \to A$, we need two properties in order to define a connected Galois covering of $\mathrm{End}_A(T)$:

1. T is of the first with respect to F,
2. $F.T$ is a basic $\mathcal{C}$-module.

For the first property, we use the following lemma which is due to W. Crawley-Boevey. I acknowledge B. Keller and C. Geiss who gave me the details about this lemma.

Lemma 6.1 *(W. Crawley-Boevey). Let $G = \mathbb{Z}/p\mathbb{Z}$ where p is the characteristic of k ($G = \mathbb{Z}$ if $char(k) = 0$). Let $\mathcal{C}$ be a locally bounded k-category and let $F : \mathcal{C} \to A$ be a Galois covering with group G. Let M be an indecomposable A-module such that $\mathrm{Ext}^1_A(M, M) = 0$. Then M is of the first kind with respect to F.*

Proof *(W. Crawley-Boevey).* Since there is no reference for this lemma, we give a sketch of the proof in the case $char(k) = 0$ (the remaining case is dealt with similarly). Thanks to [14, Section 3] (see also [12, Section 3]), the k-algebra A admits a $\mathbb{Z}$-grading and we only need to endow M with a $\mathbb{Z}$-graded A-module structure. For every $a \in A$, homogeneous of degree d, we write $\lambda_a : M \to M$ for the k-linear mapping:

$$m \mapsto d\ a\ m.$$

Using the grading on A, this defines the k-linear mapping $\lambda_a : M \to M$ for any $a \in A$. The mapping $A \to \mathrm{End}_k(M)$, $a \mapsto \lambda_a$ is a derivation, and it is inner because $\mathrm{Ext}^1_A(M, M) = 0$. So there exists $f \in \mathrm{End}_k(M)$ such that for any $m \in M$ and for any $a \in A$, homogeneous of degree d, we have:

$$d\ a\ m = f(am) - af(m).$$

For each $\lambda \in k$, denote by $M(\lambda)$ the generalised eigenspace $M(\lambda) := \{m \in M \mid (f - \lambda)^n(m) = 0$ for some $n \geqslant 0\}$ of f, so that $M = \bigoplus_{\lambda \in k} M(\lambda)$ as a k-vector space. Now, consider $\mathbb{Z}$ as a subgroup of k. If $c \in k/\mathbb{Z}$, then the subspace $M_c := \bigoplus_{\lambda \in c} M(c)$ is easily seen to be an A-submodule of M. Using the vector space decomposition $M = \bigoplus_{c \in k/\mathbb{Z}} M_c$ and the indecomposability of M, we deduce that $M = M_{c_0}$ for some $c_0 \in k/\mathbb{Z}$. It is not difficult to check that it one fixes $t \in c_0$, then the vector space decomposition $M = \bigoplus_{n \in \mathbb{Z}} M(t + n)$ endows M with a $\mathbb{Z}$-graded A-module. So the lemma is proved. □

Remark 6.2. Lemma 6.1 also holds for any group in the class of groups containing $\mathbb{Z}/p\mathbb{Z}$ and closed under direct product, under quotient and under isomorphism.

Lemma 6.3. *Let G be a finite group, let $F : A' \to A$ be a connected Galois covering with group G and let T be a basic tilting A-module of the first kind with respect to F. Then $F.T$ is a basic A'-module.*

Proof. From Lemma 1.2 applied to T, we have $\mathrm{pd}_{A'}(F.T) < \infty$. Besides, $F_\lambda F.T \simeq \bigoplus_{g \in G} T$ because T is of the first kind with respect to F, so that, for every $i \geqslant 1$:

$$\mathrm{Ext}^i_{A'}(F.T, F.T) \simeq \mathrm{Ext}^i_A(F_\lambda F.T, T) \simeq \mathrm{Ext}^i_A\left(\bigoplus_{g \in G} T, T\right) \simeq \bigoplus_{g \in G} \mathrm{Ext}^i_A(T, T) = 0.$$

Moreover, if we apply $F.$ to any finite coresolution of A in $\mathrm{add}(T)$, we deduce a finite coresolution of $A' \simeq F.A$ in $\mathrm{add}(F.T)$. Hence $F.T$ is a tilting A'-module. Now we can prove that $F.T$ is basic. Since T is basic, it is the direct sum of n indecomposable A-modules. Moreover, T is of the first kind with respect to F, so $F.T$ is the direct sum of $n.|G| = rk(K_0(A'))$ indecomposable A'-modules. Since $F.T$ is a tilting A' module, this implies that $F.T$ is basic. □

Now we can prove Proposition 1.

Proof of Proposition 1. Recall (see for example [16]) that $HH^1(kQ) = 0$ if and only if Q is a tree. Now assume that Q is not a tree. Then, A admits a connected Galois covering $F : \mathcal{C} \to A$ with group $\mathbb{Z}$. Let G be any finite cyclic group if $char(k) = 0$ and let $G = \mathbb{Z}/p\mathbb{Z}$ if $char(k) = p$ is non-zero. Let N be the kernel of the natural surjection $\mathbb{Z} \twoheadrightarrow G$. Then there exists a commutative diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[dd, "F"'] \arrow[dr, "p"] & \\ & \mathcal{C}/N, \arrow[dl, "\overline{F}"] \\ A & \end{tikzcd}$$

where $p:\mathcal{C} \to \mathcal{C}/N$ is the natural projection and $\overline{F}:\mathcal{C}/N \to A$ is a connected Galois covering with group G. Then:

- if $char(k) = 0$, then Lemma 6.1 implies that T is of the first kind with respect to F so that it is also of the first kind with respect to $\overline{F}$,
- if $char(k) = p$ is non-zero then Lemma 6.1 implies that T is of the first kind with respect to $\overline{F}$.

Thanks to Lemma 6.3 we deduce that $\overline{F}:\mathcal{C}/N \to A$ is connected Galois covering of A with group G and verifying: T is of the first kind with respect to F and $F.T$ is a basic $\mathcal{C}/N$-module. Using Remark 2.1 and Proposition 2.10, we deduce that $\mathrm{End}_A(T)$ admits a connected Galois covering with group $G \neq 1$. Hence $\mathrm{End}_A(T)$ is not simply connected. $\square$

Final remark

The Hasse diagram $\vec{\mathcal{K}}_A$ of basic tilting A-modules describes the combinatoric relations between tilting modules. When A is hereditary (that is, $A = kQ$ with Q a finite quiver with no oriented cycle) these combinatorics are also described by the cluster category $\mathcal{C}_Q$ of the quiver Q (see [11]). In particular, the indecomposable tilting objects in $\mathcal{C}_Q$ are displayed as the vertices of an unoriented graph. Since this graph is always connected (see [11, 3.5]) it is natural to ask if it is possible to remove all conditions concerning connected components in Theorem 1 and Corollary 2 (in the hereditary case). These developments will be detailed in a forthcoming text.

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